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Arbitrary unitarily invariant random matrix ensembles and supersymmetry

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Abstract

We generalize the supersymmetry method in random matrix theory to ensembles which are unitarily invariant, but otherwise arbitrary. Our exact approach extends a previous contribution in which we constructed a supersymmetric representation for the class of norm-dependent random matrix ensembles. Here, we derive a supersymmetric formulation under very general circumstances. A reduced probability density and a projector are identified that map the probability density from ordinary to superspace. Furthermore, it is demonstrated that setting up the theory in Fourier superspace has considerable advantages. General and exact expressions for the correlation functions are given. We also show how the use of hyperbolic symmetry can be circumvented in the present context in which the nonlinear σ model is not used. We construct exact supersymmetric integral representations of the correlation functions for arbitrary positions of the imaginary increments in the Green's functions.

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1. Introduction

The supersymmetry method is nowadays indispensable for the discussion of various advanced topics in the theory of disordered systems [1, 2] and it became equally important in numerous random matrix approaches to complex systems in general [3–6]. Random matrix theory (RMT) as originally formulated in ordinary space does not rely on Gaussian probability densities. It is only important that the random matrix ensembles are invariant under basis rotations and that they have a certain factorization property. Gaussian probability densities are highly convenient in calculations, but other probability densities are also possible, and some of those were already considered in the early days of RMT [7]. On the other hand, the supersymmetric formulations were constructed for Gaussian probability densities [1, 2, 8] by means of a Hubbard–Stratonovich transformation. Thus, the question arises naturally whether the Hubbard–Stratonovich transformation restricts the use of supersymmetry to the Gaussian form of the probability densities. We address this problem in the present contribution. We

will show that the supersymmetry method is not at all restricted in this way, and we will derive supersymmetric formulations of RMT for arbitrary unitarily invariant random matrix ensembles.

We focus on conceptual and structural issues. In particular, we are not aiming at asymptotic results in the inverse level number as following from the supersymmetric nonlinear σ model [1–3]. This latter approach was used in [9] to show universality for infinite level number in the case of non-Gaussian probability densities. Here, however, our goal is different: we address the full problem to achieve exact, i.e. non-asymptotic results. In a previous study [10], we presented supersymmetric representations for norm-dependent ensembles, where the probability densities are functions of the traced squared random matrices only. Although a series of interesting insights are revealed already in this case, the derivation can be done without actually employing deep features of supersymmetry. This is not so in the present contribution which aims at a general construction. The methods needed are very different from those of [10]. Here, we have to explore the algebraic structure of superspace.

One can also motivate the present investigation from the viewpoint of applications. We refer the reader to [10] and the literature quoted therein. Our goal to perform a conceptual study does not prevent us from giving general expressions for the correlation functions, but we refrain from looking too much into applications and defer this aspect to future work.

It will not be surprising for those who already have expertise in supersymmetry that a generalization as outlined above requires an analysis of convergence properties and thus leads inevitably to the issue of what kind of symmetries the theory in superspace should have. It was argued in [11] that hyperbolic symmetry, i.e. groups comprising compact and non-compact degrees of freedom, is necessary if one is to set up a nonlinear σ model in ordinary space. This line of reasoning carries over to superspace [1, 2], see also the recent review in [8]. We justify a procedure for how to avoid hyperbolic symmetry in the framework of our supersymmetric models. The necessity to introduce hyperbolic symmetry is exclusively rooted in the nonlinear σ model, not in supersymmetry as such. If one aims at exact, i.e. non-asymptotic results, compact supergroups suffice. For various reasons, including some related to convergence questions, we find it advantageous to map the theory onto Fourier superspace.

The paper is organized as follows. Having posed the problem in section 2, we generalize the Hubbard–Stratonovich transformation in section 3. In section 4, we derive the supersymmetric formulation in Fourier superspace. The correlation functions are expressed as eigenvalue integrals in section 5. Summary and conclusions are given in section 6.

2. Posing the problem

In section 2.1, the two relevant kinds of k-point correlation functions are defined and the relation to the generating functions is given. Thereby we also introduce our notation and conventions. We clarify what we mean by arbitrary unitarily invariant ensembles in section 2.2. In section 2.3, we show how different types of correlation functions can be related to each other by proper Fourier transforms.

2.1. Correlation and generating functions

The random matrix ensemble builds upon $N \times N$ Hermitian matrices H, having altogether N^2 independent matrix elements. A normalized probability density $P(H)$ assigns a statistical weight to the elements of the matrices H. As the Hermitian matrices are diagonalized by unitary matrices in $SU(N)$, the probability density $P(H)$ is said to define a unitary random matrix ensemble. No invariance property has been assumed yet. We are interested in the

k-point correlation functions

$$R_k(x_1, \ldots, x_k) = \int \mathrm{d}[H] P(H) \prod_{p=1}^{k} \mathrm{tr}\, \delta(x_p - H), \tag{1}$$

depending on the k energies $x_1, \ldots, x_k$. The δ functions are the imaginary parts of the matrix Green's functions, $\mp \mathrm{i}\pi\, \delta(x_p - H) = \mathrm{Im}(x_p \pm \mathrm{i}\varepsilon - H)^{-1}$. Here, $\mathrm{i}\varepsilon$ is an imaginary increment and the limit $\varepsilon \to 0$ is suppressed in the notation. In the supersymmetric construction to follow, it is convenient to consider the more general correlation functions which also include the real parts of the Green's functions. They are, apart from an irrelevant overall sign, given by

$$\widehat{R}_k(x_1, \ldots, x_k) = \frac{1}{\pi^k} \int \mathrm{d}[H] P(H) \prod_{p=1}^{k} \mathrm{tr}\, \frac{1_N}{x_p - \mathrm{i}L_p\varepsilon - H}. \tag{2}$$

One often wants to put the imaginary increments on different sides of the real axis. The quantities L_p which are either $+1$ or -1 determine the side of the real axis where the imaginary increment is placed. They define a metric L. The correlation function can always be expressed as derivatives of a generating function $Z_k(x + J)$ such that

$$\widehat{R}_k(x_1, \ldots, x_k) = \frac{1}{(2\pi)^k} \frac{\partial^k}{\prod_{p=1}^{k} \partial J_p} Z_k(x + J)\bigg|_{J_p = 0} \tag{3}$$

where

$$Z_k(x + J) = \int \mathrm{d}[H] P(H) \prod_{p=1}^{k} \frac{\det(H - x_p + \mathrm{i}L_p\varepsilon - J_p)}{\det(H - x_p + \mathrm{i}L_p\varepsilon + J_p)}. \tag{4}$$

We introduced source variables J_p, $p = 1, \ldots, k$ as well as the diagonal matrices $x = \mathrm{diag}(x_1, x_1, \ldots, x_k, x_k)$ and $J = \mathrm{diag}(-J_1, +J_1, \ldots, -J_k, +J_k)$. In the following, we use the shorthand notation $x_p^{\pm} = x_p - \mathrm{i}L_p\varepsilon$ and $x^{\pm} = \mathrm{diag}\big(x_1^{\pm}, x_1^{\pm}, \ldots, x_k^{\pm}, x_k^{\pm}\big)$. The product of the differentials of all independent matrix elements is the volume element $\mathrm{d}[H]$. We use the notation and the conventions of [12–14]. The normalization $Z_k(x) = 1$ at $J = 0$ follows immediately from the definition (4).

We wish to study whether the generating function can be represented as an integral of the form

$$Z_k(x + J) = \int \mathrm{d}[\sigma] Q(\sigma) \mathrm{detg}^{-N}(\sigma - x^{\pm} - J), \tag{5}$$

where σ is a $2k \times 2k$ supermatrix with Hermitian or related symmetries, and where detg denotes the superdeterminant. If such a representation can be shown to exist, the question arises whether the probability density $Q(\sigma)$ in superspace can be obtained in a unique way from the probability density $P(H)$ in ordinary space.

2.2. *Unitarily invariant probability densities*

For the important class of norm-dependent ensembles, i.e. ensembles defined by a probability density depending exclusively on $\mathrm{tr}\, H^2$, such a unique construction is indeed possible and was performed in [10]. Here, we tackle the problem of arbitrary unitarily invariant probability densities $P(H)$. We recall that a probability density must be normalizable and positively semi-definite. The term 'arbitrary' has to be understood as excluding those functions $P(H)$ which would lead to a divergent integral (4). By 'unitarily invariant' we mean that the probability density has the property

$$P(H) = P\big(U_0 H U_0^{\dagger}\big) = P(E), \tag{6}$$

where U_0 is any fixed matrix in $SU(N)$ and where $E = \mathrm{diag}(E_1, \ldots, E_N)$ is the diagonal matrix of the eigenvalues of H. Although it is obvious, we underline that this includes invariance under permutations of the vectors defining the basis in which H is written down and also invariance under permutations of the eigenvalues. Hence, the probability density $P(H)$ should depend only on matrix invariants, such as $\mathrm{tr}\, H^m$ where m is real and positive. Anticipating the later discussion, we already now mention that this requirement is a most natural one in view of the general character of the supersymmetry method. The strength of this method is rooted in the drastic reduction of degrees of freedom, i.e. of the number of integration variables, when an integral over the $N \times N$ matrix H is identically rewritten as an integral over the $2k \times 2k$ matrix σ. Thus, supersymmetry removes a certain redundancy. The invariance requirement implies precisely this redundancy which the supersymmetry method needs. We will show that this holds for arbitrary unitarily invariant probability densities $P(H)$.

2.3. Mutual relations between the different correlation functions

We wish to address the correlation functions (1) and (2) for finite level number N, we are not aiming at an asymptotic discussion. If a saddlepoint approximation leading to a nonlinear σ model as in [1, 3, 11] is the method of choice to study a certain physics problem, one performs precisely such an asymptotic expansion in $1/N$. This is not what we are going to do in the present contribution. Admittedly, our goal to address the problem exactly for finite N renders our task mathematically demanding, because we have to solve certain group integrals. One the other hand, luckily and at first sight paradoxically, this goal allows us to circumvent the introduction of hyperbolic symmetry, which is a deeply rooted, non-trivial feature of the nonlinear σ model [1, 3, 8, 11]. Hyperbolic symmetry means that the ensuing supersymmetric representation of the random matrix model must involve non-compact groups to make the integrals convergent. This is inevitable if the imaginary increments of the energies lie on different sides of the real axis. However, if they lie on the same side, no hyperbolic symmetry occurs and all groups are compact. This facilitates the supersymmetric treatment tremendously.

We now argue that the correlation functions (1) of the imaginary parts can be recovered from the more general correlation functions (2) that are suited for the supersymmetric treatment, even if all imaginary increments lie on the same side of the real axis. We choose $L_p = +1$ for all $p = 1, \ldots, k$. Upon Fourier transforming the correlation functions (2), we obtain the k-point correlations in the domain of the times t_p, $p = 1, \ldots, k$,

$$\begin{aligned}\widehat{r}_k(t_1, \ldots, t_k) &= \frac{1}{\sqrt{2\pi}^k} \int_{-\infty}^{+\infty} \mathrm{d}x_1 \exp(\mathrm{i}t_1 x_1) \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_k \exp(\mathrm{i}t_k x_k) \widehat{R}_k(x_1, \ldots, x_k) \\ &= (\mathrm{i}2)^k \prod_{p=1}^{k} \Theta(t_p) \exp(-\varepsilon t_p) r_k(t_1, \ldots, t_k) \end{aligned} \tag{7}$$

with

$$r_k(t_1, \ldots, t_k) = \frac{1}{\sqrt{2\pi}^k} \int \mathrm{d}[H] P(H) \prod_{p=1}^{k} \mathrm{tr} \exp(\mathrm{i}Ht_p). \tag{8}$$

Importantly, this latter k-point correlation function $r_k(t_1, \ldots, t_k)$ in time domain is precisely the Fourier transform of the correlation function (1). It is well defined on the entire real axes of all its arguments t_p. The inverse transform yields

$$R_k(x_1, \ldots, x_k) = \frac{1}{\sqrt{2\pi}^k} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \exp(-\mathrm{i}x_1 t_1) \cdots \int_{-\infty}^{+\infty} \mathrm{d}t_k \exp(-\mathrm{i}x_k t_k) r_k(t_1, \ldots, t_k). \tag{9}$$

Based on this observation, we will pursue the following strategy in later Sections of this contribution: We perform exact manipulations of the correlations $\widehat{R}_k(x_1, \ldots, x_k)$ with $L_p = +1$, $p = 1, \ldots, k$, or of their generating functions, respectively. Having obtained the appropriate supersymmetric representation, we Fourier transform it into the time domain and find $\widehat{r}_k(t_1, \ldots, t_k)$. In this expression, we then identify the supersymmetric representation of the correlation functions $r_k(t_1, \ldots, t_k)$. Upon backtransforming we arrive at the desired supersymmetric representation for the correlation functions $R_k(x_1, \ldots, x_k)$.

We can even extend the line of arguing. Once we have found the supersymmetric representation of $r_k(t_1, \ldots, t_k)$, we can construct the one of $\widehat{R}_k(x_1, \ldots, x_k)$ for any arbitrary choice of the quantities $L_p = \pm 1$ by calculating

$$\widehat{R}_k(x_1, \ldots, x_k) = \frac{1}{\sqrt{2\pi}^k} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \exp(-\mathrm{i}x_1 t_1) \cdots \int_{-\infty}^{+\infty} \mathrm{d}t_k \exp(-\mathrm{i}x_k t_k) \times \prod_{p=1}^{k} (\mathrm{i}^{L_p} 2)\Theta(L_p t_p) \exp(-\varepsilon L_p t_p) r_k(t_1, \ldots, t_k). \tag{10}$$

In this manner, we will obtain supersymmetric integral representations for all correlation functions (2) where the imaginary increments lie independently of each other on either side of the real axes, without introducing hyperbolic symmetry.

3. Generalized Hubbard–Stratonovich transformation

To carry out the programme outlined in section 2, we have to generalize the procedure referred to as Hubbard–Stratonovich transformation. In section 3.1, we Fourier transform the probability density. An algebraic duality between matrix structures is uncovered in section 3.2, and explored further in section 3.3, where spectral decompositions of the matrices involved are performed. Although our main interest are the correlation functions where all L_p are equal, we make these latter steps for an arbitrary metric L. We do so, because we find it worthwhile to document how natural the duality is even for a general metric. Moreover, it allows us to clearly identify the point where a general metric would require a much involved discussion of hyperbolic symmetry—which we then avoid by setting $L_p = +1$ for all $p = 1, \ldots, k$. In section 3.4, we construct the probability density in superspace. A reduced probability density is introduced in section 3.5. We derive a generalized transformation formula and the corresponding generating function in sections 3.6 and 3.7, respectively. In section 3.8, the norm-dependent ensembles are discussed as a simple example.

3.1. Fourier transform of the probability density

The determinants in the generating function (4) are written as Gaussian integrals, those in the denominator as integrals over k vectors z_p, $p = 1, \ldots, k$ with N complex commuting elements each, and those in the numerator over k vectors ζ_p, $p = 1, \ldots, k$ with N complex anticommuting elements each. Again omitting irrelevant phase factors, we have

$$Z_k(x + J) = \int \mathrm{d}[H] P(H) \prod_{p=1}^{k} \int \mathrm{d}[z_p] \exp\left(\mathrm{i}L_p z_p^\dagger (H - x_p + \mathrm{i}L_p \varepsilon + J_p) z_p\right) \times \int \mathrm{d}[\zeta_p] \exp\left(\mathrm{i}\zeta_p^\dagger (H - x_p + \mathrm{i}L_p \varepsilon - J_p)\zeta_p\right), \tag{11}$$

where $\mathrm{d}[z_p]$ and $\mathrm{d}[\zeta_p]$ denote the products of the independent differentials. To ensure convergence of the integrals over the commuting variables, the quantities L_p are inserted

in front of the bilinear forms in the exponent. This is not needed in the integrals over the anticommuting variables because they are always convergent. We order the quantities L_p in the metric tensor $L = \mathrm{diag}(L_1, 1, \ldots, L_k, 1)$. Using the identities

$$z_p^\dagger H z_p = \mathrm{tr}\, H z_p z_p^\dagger \qquad \text{and} \qquad \zeta_p^\dagger H \zeta_p = -\,\mathrm{tr}\, H \zeta_p \zeta_p^\dagger, \tag{12}$$

the average over H in equation (11) can be written as the Fourier transform

$$\Phi(K) = \int \mathrm{d}[H] P(H) \exp(\mathrm{i}\,\mathrm{tr}\, HK) \tag{13}$$

of the probability density. The Fourier variable is the matrix

$$K = \sum_{p=1}^{k} L_p z_p z_p^\dagger - \sum_{p=1}^{k} \zeta_p \zeta_p^\dagger. \tag{14}$$

The function $\Phi(K)$ is referred to as characteristic function. The definition (13) of the Fourier transform is the one mostly used in the statistics literature. It guarantees that $\Phi(0) = 1$, directly reflecting the normalization of $P(H)$. The definition of the Fourier transform in section 2.3 follows the 'symmetric convention' in which the same factor of $1/\sqrt{2\pi}$ appears in the transform and in its inverse.

Up to now, all steps were exactly as in the case of a Gaussian probability density $P(H)$. In the Gaussian case, one can now do the integral (13) explicitly and one obtains a Gaussian form for the characteristic function $\Phi(K)$. Here we consider a general unitarily invariant $P(H)$. Of course, we must assume that the Fourier transform exists, i.e. that $P(H)$ is absolutely integrable or, even better, that it is a Schwartz function. Absolute integrability is guaranteed by the fact that $P(H)$ is a probability density, implying that it is positively semi-definite and normalized. However, we also must assume that the integrals over the vectors z_p converge after doing the Fourier transform. The integrals over the vectors ζ_p can never cause convergence problems. In the Gaussian case, all those convergence issues have been carefully discussed in [11], a recent review is given in [8]. In the general unitarily invariant case, we have no other choice than to implicitly exclude those probability density $P(H)$ which would cause convergence problems, assuming that all integrals in the sequel converge. We will come back to this point later.

It is easy to see that the unitary invariance of $P(H)$ also implies the unitary invariance of $\Phi(K)$. The matrix K is Hermitian, $K^\dagger = K$. This is so for all choices $L_p = \pm 1$ of the metric elements. As the entries of K are commuting variables, we may conclude that K can be diagonalized:

$$K = \widetilde{V}\widetilde{Y}\widetilde{V}^\dagger \qquad \text{such that} \qquad K\widetilde{V}_n = \widetilde{Y}_n \widetilde{V}_n. \tag{15}$$

Here $\widetilde{V}$ is in $SU(N)$ and contains the eigenvectors $\widetilde{V}_n$ as columns, $\widetilde{V} = [\widetilde{V}_1 \cdots \widetilde{V}_N]$. Moreover, $\widetilde{Y} = \mathrm{diag}(\widetilde{Y}_1, \ldots, \widetilde{Y}_N)$ is the diagonal matrix containing the eigenvalues $\widetilde{Y}_n, n = 1, \ldots, N$ of K. The unitary invariance of $P(H)$ and the invariance of the measure $\mathrm{d}[H]$ allow one to absorb $\widetilde{V}$ such that the characteristic function $\Phi(K)$ depends only on $\widetilde{Y}$:

$$\Phi(K) = \int \mathrm{d}[H] P(H) \exp(\mathrm{i}\,\mathrm{tr}\, H\widetilde{Y}) = \Phi(\widetilde{Y}). \tag{16}$$

In other words, $\Phi(K)$ is a unitarily invariant function too.

3.2. Underlying algebraic duality

The merit of the Hubbard–Stratonovich transformation in the supersymmetry method is the drastic reduction in the number of degrees of freedom. This is rooted in a duality between

matrices in ordinary and superspace. We uncover this duality and cast it into a form which allows a straightforward generalization of the previous discussion for a Gaussian probability density. We define the $N \times 2k$ rectangular matrix

$$A = [z_1 \cdots z_k \zeta_1 \cdots \zeta_k]\,. \tag{17}$$

Although it contains commuting and anticommuting entries, A is not a supermatrix of the type commonly appearing in the framework of the supersymmetry method [1, 2]. Nevertheless, this matrix will play a crucial role in the following. Its Hermitian conjugate is the $2k \times N$ rectangular matrix

$$A^\dagger = \begin{bmatrix} z_1^\dagger \\ \vdots \\ z_k^\dagger \\ -\zeta_1^\dagger \\ \vdots \\ -\zeta_k^\dagger \end{bmatrix}. \tag{18}$$

The inclusion of the minus signs is necessary to be consistent with the conventions in [3, 12–14]. It ensures that we have $(A^\dagger)^\dagger = A$.

We note that the boson–fermion block notation [3] is used in the definition (17), which differs from the pq block notation [3] employed when defining the supermatrices L, x and J, as well as implicitly σ in section 2. In the boson–fermion block notation, first all commuting and then all anticommuting variables (or vice versa) are collected in a supervector. Hence, the supermatrices which linearly transform those vectors consist of rectangular blocks of commuting or anticommuting variables, in the case of σ the blocks are $k \times k$ square. The pq notation is obtained by simply reordering the basis. One collects the commuting and anticommuting variables corresponding to each energy index $p = 1, \ldots, k$, such that every supermatrix is written as a $k \times k$ ordinary matrix with 2×2 supermatrix elements assigned to each index pair (p, q). While the latter notation was handy when introducing the generating function in section 2, it is more convenient for the present discussion to use the boson–fermion block notation. In particular, the metric then reads $L = \mathrm{diag}(L_1, \ldots, L_k, 1, \ldots, 1)$.

The Hermitian $N \times N$ matrix K defined in equation (14) can be written as the matrix product

$$K = ALA^\dagger = (AL^{1/2})(L^{1/2}A^\dagger). \tag{19}$$

There exists a natural dual matrix to K, found by interchanging the order of the matrices in equation (19). It is the $2k \times 2k$ matrix

$$B = (L^{1/2}A^\dagger)(AL^{1/2}) = L^{1/2}A^\dagger AL^{1/2}, \tag{20}$$

where

$$A^\dagger A = \begin{bmatrix} z_1^\dagger z_1 & \cdots & z_1^\dagger z_k & z_1^\dagger \zeta_1 & \cdots & z_1^\dagger \zeta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ z_k^\dagger z_1 & \cdots & z_k^\dagger z_k & z_k^\dagger \zeta_1 & \cdots & z_k^\dagger \zeta_k \\ -\zeta_1^\dagger z_1 & \cdots & -\zeta_1^\dagger z_k & -\zeta_1^\dagger \zeta_1 & \cdots & -\zeta_1^\dagger \zeta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ -\zeta_k^\dagger z_1 & \cdots & -\zeta_k^\dagger z_k & -\zeta_k^\dagger \zeta_1 & \cdots & -\zeta_k^\dagger \zeta_k \end{bmatrix}. \tag{21}$$

While $K = ALA^\dagger$ is an ordinary matrix, $A^\dagger A$ and $= L^{1/2}A^\dagger AL^{1/2}$ are supermatrices. Moreover, K is Hermitian for all choices of the metric L, i.e. for every combination $L_p = \pm 1$, but B is in general not Hermitian because some entries of the metric are imaginary, $L_p^{1/2} = \mathrm{i}$. The supermatrix $A^\dagger A$, however, is Hermitian.

Interestingly, the duality between the matrices K and B also implies the equality of invariants involving the traces according to

$$\mathrm{tr}\, K^m = \mathrm{trg}\, B^m, \tag{22}$$

for every non-zero, positive integer m. This generalizes the case of a Gaussian probability density where the need to discuss this equality occurs only for $m = 2$. As the equality is not completely trivial due to the presence of anticommuting variables, equation (22) is proven in appendix A.

3.3. Eigenvalues and eigenvectors of the dual matrices

Our way of formulating the algebraic duality is most helpful for the spectral decomposition in ordinary and superspace. We write the eigenvalue equation for the matrix K as

$$KV_n = Y_n V_n, \tag{23}$$

with N eigenvalues Y_n and corresponding eigenvectors $V_n, n = 1, \ldots, N$. We will now construct them in such a manner that they are closely related to, but slightly different from the eigenvalues $\widetilde{Y}_n$ and eigenvectors $\widetilde{V}_n$ following from the definition (15). We emphasize that $\widetilde{V}$ is in the standard $SU(N)$ while our construction of the eigenvectors V_n will not impose this condition from the beginning. The connection between the two definitions will be clarified in detail later on. We employ the $2k$ component supervectors

$$w_n = \begin{bmatrix} w_{n11} \\ \vdots \\ w_{nk1} \\ w_{n12} \\ \vdots \\ w_{nk2} \end{bmatrix}. \tag{24}$$

There are two distinct representations of these supervectors. In the first one, the elements w_{npj} are commuting if $j = 1$ and anticommuting if $j = 2$, in the second representation it is the other way around. We make the ansatz

$$V_n = AL^{1/2}w_n = \sum_{p=1}^{k} z_p L_p^{1/2} w_{np1} + \sum_{p=1}^{k} \zeta_p w_{np2} \tag{25}$$

for the eigenvectors. It is convenient to multiply the coefficients w_{npj} from the right to avoid some cumbersome signs if w_{np2} are anticommuting and appear together with the vectors ζ_p. We plug the ansatz (25) into the eigenvalue equation (23) and find

$$\begin{aligned} KV_n &= ALA^\dagger AL^{1/2}w_n = AL^{1/2}Bw_n \\ Y_n V_n &= Y_n AL^{1/2}w_n = AL^{1/2}Y_n w_n, \end{aligned} \tag{26}$$

which yields $AL^{1/2}(Bw_n - Y_n w_n) = 0$. Hence, we conclude that the eigenvalue equation

$$Bw_n = Y_n w_n \tag{27}$$

holds if the eigenvalue equation (23) is valid and if the eigenvectors V_n have the form (25). There is a duality: the eigenvalues Y_n of K to the eigenvectors V_n in the form (25) are also eigenvalues of B to the eigenvectors w_n.

The fact that the eigenvectors w_n of the supermatrix B belong to one distinct representation as discussed below equation (24) implies that there are two types of eigenvalues corresponding to these representations. We denote the k eigenvalues associated with the first representation by $y_{p1} = Y_p$, $p = 1, \ldots, k$ and the k eigenvalues associated with the second one by $y_{p2} = Y_{k+p}$, $p = 1, \ldots, k$, respectively. Moreover, not all eigenvectors V_n of K can have the form (25) if the vector w_n is required to be eigenvector of B at the same time. This is so, because K and B have different dimensions $N \times N$ and $2k \times 2k$, respectively. In all applications of RMT and supersymmetry, the level number N is large, such that we may safely assume $N > 2k$. The matrix B has $2k$ eigenvalues. Thus, the duality uncovered above only makes a statement about $2k$ out of the N eigenvalues of K. Importantly, the remaining eigenvalues of K are zero, because K is built upon $2k$ dyadic matrices. Hence, we have

$$Y_n = \begin{cases} y_{p1} & \text{for } n = p,\ p = 1, \ldots, k \\ y_{p2} & \text{for } n = p + k,\ p = 1, \ldots, k \\ 0 & \text{for } n = 2k + 1, \ldots, N, \end{cases} \tag{28}$$

if $N > 2k$. As K is an ordinary Hermitian matrix, we know that the eigenvectors V_n to the zero eigenvalues can be chosen orthogonal with each other and with those to the non-zero eigenvalues.

The presence of anticommuting variables often implies curious features. Here, it is a sign switch between the eigenvalues $\widetilde{Y}_n$ and Y_n according to the definitions (23) and (15). It occurs for those eigenvalues which are related to the second representation of the supervectors w_n,

$$\widetilde{Y}_n = \begin{cases} +Y_n = +y_{p1} & \text{for } n = p,\ p = 1, \ldots, k \\ -Y_n = -y_{p2} & \text{for } n = p + k,\ p = 1, \ldots, k, \end{cases} \tag{29}$$

the other eigenvalues are of course zero as well. Wei [16] observed that such a sign switch is indeed necessary for consistency reasons. This is, for example, seen from equation (22) for $m = 1$.

The line of reasoning given above is supplemented with various details in appendix B, including a derivation of the relations between the eigenvalues Y_n and $\widetilde{Y}_n$ and of the eigenvectors V_n and $\widetilde{V}_n$.

3.4. Probability density in superspace

The characteristic function $\Phi(K)$ of the probability density $P(H)$ is according to equation (16) unitarily invariant, $\Phi(K) = \Phi(\widetilde{Y})$. Furthermore, by virtue of the previous discussion we may view it as function of the eigenvalues of the supermatrix B, such that we arrive at the chain of equalities

$$\Phi(K) = \Phi(\widetilde{Y}) = \Phi(y) = \Phi(B). \tag{30}$$

This crucial observation identifies Φ as an invariant function in two different spaces, in ordinary space depending on the $N \times N$ matrix K and in superspace, depending on the dual $2k \times 2k$ matrix B. It is interesting to note that, if $\Phi(K)$ is a function of all invariants $\operatorname{tr} K^m$ with $m = 1, 2, 3, \ldots$, we may conclude from the equality (22) the identity

$$\Phi(\operatorname{tr} K, \operatorname{tr} K^2, \operatorname{tr} K^3, \ldots) = \Phi(\operatorname{trg} B, \operatorname{trg} B^2, \operatorname{trg} B^3, \ldots), \tag{31}$$

implying that the form of $\Phi(K)$ as a function of those invariants fully carries over to superspace. It is not obvious whether the characteristic function can contain invariants $\operatorname{tr} K^m$ with non-integer m, see the discussion in appendix B. We had to assume integer m when deriving the equality (22) in appendix A. If invariants $\operatorname{tr} K^m$ with non-integer m can be present, the chain

of equalities (30) would be more general than equation (31). We note that the determinant $\det K$ is trivially excluded from the relevant invariants, because we know from the previous section that K has zero eigenvalues if $N > 2k$. In the following, we will base our derivation on the chain of equalities (30). However, in the case of ambiguities, we find it always safer to resort to the slightly less general formula (31).

We restrict the further discussion to the case that all imaginary increments of the energies lie on the same side of the real axis. Hence we choose the metric

$$L = +1_{2k}. \tag{32}$$

This choice implies that the supermatrix B becomes Hermitian, $B^\dagger = B$, and the symmetry group is $U(k/k)$, the unitary supergroup in k bosonic and k fermionic dimensions. For a general metric, the corresponding relation reads $B^\dagger = LBL$. The symmetry group is pseudounitary, i.e. the matrices of the defining representation satisfy $w^\dagger L w = L$. This hyperbolic symmetry involves non-compact degrees of freedom. The situation was analysed in detail in [11] and [1] for the nonlinear σ model in ordinary and in superspace, respectively. The proper, convergence ensuring integration manifolds of the Hubbard–Stratonovich fields, corresponding to the matrices σ in the present case, was constructed. It seems to us that the Gaussian form of the probability densities $P(H)$ in these investigations was somehow important for this construction. Here, however, we study arbitrary unitarily invariant probability densities $P(H)$. We did not succeed in extending the line of reasoning in [1, 11] to such general $P(H)$, even though we strongly believe that this should be possible. Nevertheless, this does not cause a problem in view of what we are aiming at, because we can proceed as outlined in section 2.3. All issues related to convergence can be dealt with much easier if the choice (32) is made. That this works fine in the case of a Gaussian $P(H)$ was already demonstrated in [12].

As $\Phi(K)$ is the characteristic function of $P(H)$ in ordinary space, the chain of equalities (30) naturally suggests to interpret $\Phi(B)$ as a characteristic function in superspace. To this end, we introduce a probability density $Q(\sigma)$ depending on a $2k \times 2k$ supermatrix σ whose Fourier transform is $\Phi(B)$. However, there is a subtle point to which we have to pay attention. The symmetries of B dictate to a large extent what the symmetries of σ have to be. As B is a Hermitian supermatrix, σ ought to be a Hermitian supermatrix as well. Nevertheless, a Wick type of rotation was applied in the case of Gaussian probability densities which provides all elements in the fermion–fermion block of σ with an imaginary unit i [1, 3]. This modification is needed to solve a convergence problem too. It makes the expression $\mathrm{trg}\,\sigma^2$ positive semi-definite, and thus the integrals over the Gaussian probability density $Q(\sigma) \sim \exp(-\mathrm{trg}\,\sigma^2)$ convergent. It should be stressed that the Wick type of rotation is, in general, not a unique procedure. It must implicitly depend on the specific form of the probability density $P(H)$. As we want to include the Gaussian as a special case in our considerations, we also introduce the same Wick type of rotation as in [1, 3] in the $2k \times 2k$ supermatrix σ. Other choices might be necessary in specific applications.

The entries of the matrix B can be modified according to our present Wick type of rotation by multiplying the vectors ζ_p containing the anticommuting variables with factors of $\sqrt{\mathrm{i}}$, if one wishes, but we do not do that here. The diagonalization of the matrix σ can be written as

$$\sigma = usu^\dagger \qquad \text{with} \quad s = \mathrm{diag}(s_{11}, \ldots, s_{k1}, \mathrm{i}s_{12}, \ldots, \mathrm{i}s_{k2}), \tag{33}$$

where all eigenvalues s_{pj} are real. The Wick type of rotation multiplies the eigenvalues s_{p2} with an imaginary unit. Thus, u is in the unitary supergroup $U(k/k)$, without any modification of its matrix elements. We also introduce a $2k \times 2k$ supermatrix ρ with the same symmetries as σ and with the diagonalization

$$\rho = vrv^\dagger \qquad \text{with} \quad r = \mathrm{diag}(r_{11}, \ldots, r_{k1}, \mathrm{i}r_{12}, \ldots, \mathrm{i}r_{k2}), \tag{34}$$

where v is in the unitary supergroup $U(k/k)$ as well.

To define the probability density $Q(\sigma)$ in superspace, we first formally replace B with ρ in the characteristic function,

$$\Phi(B) \longrightarrow \Phi(\rho), \tag{35}$$

on the level of equation (31), this means

$$\Phi(\mathrm{trg}\, B, \mathrm{trg}\, B^2, \mathrm{trg}\, B^3, \ldots) \longrightarrow \Phi(\mathrm{trg}\, \rho, \mathrm{trg}\, \rho^2, \mathrm{trg}\, \rho^3, \ldots). \tag{36}$$

This fixes our way of doing the Wick type of rotation. We can then in a well-defined manner introduce the probability density in superspace through the Fourier integral

$$\int \mathrm{d}[\sigma] Q(\sigma) \exp(\mathrm{i}\, \mathrm{trg}\, \sigma\rho) = \Phi(\rho), \tag{37}$$

or, as $\Phi(r)$ is invariant, we have equivalently

$$\int \mathrm{d}[\sigma] Q(\sigma) \exp(\mathrm{i}\, \mathrm{trg}\, \sigma r) = \Phi(r). \tag{38}$$

We recall that the matrix elements of B in the fermion–fermion block are the scalar products $\zeta_p^\dagger \zeta_q$ and thus nilpotent variables. This implies that $\Phi(B)$ as a function of these variables is a finite power series. However, when replacing B with ρ we continue this power series to an infinite one. It is important to realize that this step is not problematic at all, because $\Phi(B) = \Phi(K)$ results from the Fourier transform of $P(H)$. To illustrate the feasibility of this continuation, we refer to the cases covered by equation (31).

When writing the expression $\mathrm{trg}\, \sigma\rho$, one sees that the imaginary units due to the Wick type of rotation in the fermion–fermion blocks nicely combine to -1 such that the whole expression $\mathrm{trg}\, \sigma\rho$ is real. This is of course necessary to make the Fourier transform well defined. The inverse of the Fourier transform (37) reads

$$Q(\sigma) = 2^{2k(k-1)} \int \mathrm{d}[\rho] \Phi(\rho) \exp(-\mathrm{i}\, \mathrm{trg}\, \sigma\rho). \tag{39}$$

We note that the prefactor $2^{2k(k-1)}$ does not involve π, because we have the same number of commuting and anticommuting variables. Due to the invariance of the measure $\mathrm{d}[\rho]$, the unitary invariance of the characteristic function $\Phi(\rho)$ gives with equation (39) directly the same property for the probability density, such that

$$Q(\sigma) = Q(s). \tag{40}$$

The unitary invariance of $P(H)$ implies the corresponding feature for $Q(\sigma)$.

There is a good reason why we defined $Q(\sigma)$ as above. Nevertheless, what we need now to carry through our construction, is the integral representation

$$\int \mathrm{d}[\sigma] Q(\sigma) \exp(\mathrm{i}\, \mathrm{trg}\, \sigma B) = \Phi(B) \tag{41}$$

of the characteristic function considered as a function of B. At first sight, there is a problem, because $\mathrm{trg}\, \sigma B$ is not real anymore. As the imaginary unit is present in the fermion–fermion block of σ, but absent in that of B, the Fourier integral (41) seems ill defined. However, as argued above, one can also Wick-rotate the relevant elements of B. Even if one chooses not to do that, everything is under control, because the matrix elements $\zeta_p^\dagger \zeta_q$ of B in the fermion–fermion block are nilpotent. The corresponding expressions in $\exp(\mathrm{i}\, \mathrm{trg}\, \sigma B)$ consist of a finite number of terms, and no convergence problem for the σ integration can occur.

3.5. Reduced probability density

Employing the results (28) and (29) for the eigenvalues of the matrix K, we have

$$\mathrm{tr}\, H\widetilde{Y} = \sum_{n=1}^{N} H_{nn}\widetilde{Y}_n = \sum_{p=1}^{k} H_{pp}y_{p1} - \sum_{p=1}^{k} H_{(k+p)(k+p)}y_{p2}. \tag{42}$$

Inserting this into equation (16), using the chain of equalities (30) and carrying out the replacement (35), we find

$$\Phi(r) = \int \mathrm{d}[H]P(H)\exp(\mathrm{i}\,\mathrm{trg}\,hr). \tag{43}$$

Here, we introduced the diagonal matrix

$$h = \mathrm{diag}(H_{11},\ldots,H_{kk},\mathrm{i}H_{(k+1)(k+1)},\ldots,\mathrm{i}H_{(2k)(2k)}), \tag{44}$$

where the imaginary units are inserted for convenience. One concludes that only a reduced probability density enters the further consideration which depends exclusively on the $2k$ diagonal elements h of H appearing in equation (42). While this is indeed correct, a naive definition of the type $P^{(\mathrm{red})}(h) = \int \mathrm{d}[H/h]P(H)$ for the reduced probability density turns out to be problematic. It would not preserve any information about supersymmetry, although we know that this structure is essential. The proper way of defining $P^{(\mathrm{red})}(h)$ is by explicitly using the characteristic function which, as shown in detail, contains all the information about supersymmetry. The above-mentioned naive definition serves as a guideline to obtain from equation (43) the meaningful expression

$$\Phi(r) = \int \mathrm{d}[h]P^{(\mathrm{red})}(h)\exp(\mathrm{i}\,\mathrm{trg}\,hr), \tag{45}$$

which implicitly defines the reduced probability density as the Fourier transform of the characteristic function $\Phi(r)$ in the space of the $2k$ coordinates h. Thus, upon backtransforming, we arrive at

$$P^{(\mathrm{red})}(h) = \frac{1}{(2\pi)^{2k}}\int \mathrm{d}[r]\exp(-\mathrm{i}\,\mathrm{trg}\,hr)\Phi(r). \tag{46}$$

The coordinates h and r are here viewed as describing a flat space. This is very different from Fourier transforms in curved space, where the eigenvalues r function in a direct sense as the radial coordinates of a Hermitian supermatrix ρ. A proper definition of the diagonal matrix h made it possible to employ the supertrace in equations (45) and (46). The necessity to be so careful when defining the reduced probability density $P^{(\mathrm{red})}(h)$ will be illustrated by an example later on. Unfortunately, it is inevitable to include material to be derived in the ensuing sections when assembling this example. We are thus forced to postpone its discussion, it will be presented in appendix F.

In all these considerations above, we have not used equation (31). If only invariants of the form $\mathrm{trg}\,\rho^m = \mathrm{trg}\,r^m$ with integer m enter the characteristic function, the defining expression (46) implies that the reduced probability density $P^{(\mathrm{red})}(h)$ is as well only a function of the corresponding quantities $\mathrm{trg}\,h^m$. This is easily seen by permuting any pair of the diagonal elements either within $H_{11},\ldots,H_{kk}$ or within $H_{(k+1)(k+1)},\ldots,H_{(2k)(2k)}$ in $P^{(\mathrm{red})}(h)$. By exchanging the integration variables in $r_{11},\ldots,r_{k1}$ or $r_{12},\ldots,r_{k2}$ accordingly, all such permutations can be absorbed again. Importantly, permutations mixing the groups $H_{11},\ldots,H_{kk}$ and $H_{(k+1)(k+1)},\ldots,H_{(2k)(2k)}$ cannot be redone in general. This is so because of the imaginary units. There are only some exceptional cases, for example if $\Phi(r)$ is only a function of $\mathrm{trg}\,r^2$. The exclusive dependence of $P^{(\mathrm{red})}(h)$ on the quantities $\mathrm{tr}\,\mathrm{g}\,h^m$ is an invariance property of a rather unusual form, because the entries of h are not eigenvalues.

3.6. Generalized transformation formula

We plug the characteristic function as given by equation (45) into the inverse Fourier transform (39) and find

$$Q(\sigma) = 2^{2k(k-1)} \int \mathrm{d}[\rho] \exp(-\mathrm{i}\,\mathrm{trg}\,\sigma\rho) \int \mathrm{d}[h] P^{(\mathrm{red})}(h) \exp(\mathrm{i}\,\mathrm{trg}\,hr). \tag{47}$$

Assuming that the order of integrations may be interchanged, we arrive at the generalized transformation formula

$$Q(\sigma) = \int \mathrm{d}[h] P^{(\mathrm{red})}(h) \chi(\sigma, h). \tag{48}$$

The function

$$\chi(\sigma, h) = 2^{2k(k-1)} \int \mathrm{d}[\rho] \exp(\mathrm{i}\,\mathrm{trg}(hr - \sigma\rho)) \tag{49}$$

is a projector which is related to, but different from a δ function. It might look surprising that the integrand contains the full matrix ρ as well as its eigenvalue matrix r, but recalling the derivation, this is rather natural. The term $\exp(\mathrm{i}\,\mathrm{trg}\,hr)$ stems from the Fourier transform of the probability density $P(H)$ in ordinary space. Although it is conveniently written in a supersymmetric notation, it is exclusively rooted in ordinary space. Thus, anticommuting variables may only implicitly be present, which makes it plausible that r appears, but not the full ρ.

The projector satisfies the important normalization property

$$\int \mathrm{d}[\sigma] \chi(\sigma, h) = \int \mathrm{d}[\rho] \delta^{(4k^2)}(\rho) \exp(\mathrm{i}\,\mathrm{trg}\,hr) = 1, \tag{50}$$

where $\delta^{(4k^2)}(\rho)$ is the product of the δ functions of all $4k^2$ independent matrix elements in the supermatrix ρ. This then gives directly the normalization

$$\int \mathrm{d}[\sigma] Q(\sigma) = \int \mathrm{d}[h] P^{(\mathrm{red})}(h) \int \mathrm{d}[\sigma] \chi(\sigma, h) = \int \mathrm{d}[h] P^{(\mathrm{red})}(h) = 1 \tag{51}$$

of the probability density in superspace. As one should expect, the normalization of $P(H)$, which implies the normalization of $P^{(\mathrm{red})}(h)$, yields the normalization of $Q(\sigma)$. One is tempted to conclude that this feature wraps up the whole convergence discussion if the choice (32) has been made. Such an interpretation is corroborated by the character of the projector $\chi(\sigma, h)$. As it shares features with a δ function, it certainly improves the convergence properties when appearing in an integral. Nevertheless, this optimism regarding the convergence properties comes to terms when considering the complexity of all intermediate steps. Unfortunately, it prevents us at present from providing the impression stated above with more mathematical substance for a general $P(H)$. One possible problem is related to the Wick type of rotation. All invariants $\mathrm{tr}\,H^{2m} = \mathrm{tr}\,E^{2m}$ are positive semi-definite for all integer m. This is clearly not so for the corresponding invariants $\mathrm{trg}\,\sigma^{2m} = \mathrm{trg}\,s^{2m}$, where we have positive semi-definiteness only for odd integers m. This does not inevitably lead to difficulties, because a term $\exp(-\mathrm{tr}\,H^{2m})$ in $P(H)$ is not necessarily mapped onto its analogue $\exp(-\mathrm{trg}\,\sigma^{2m})$ in $Q(\sigma)$, but it illustrates, first, at which points problems could arise and, second, the need to adjust the Wick type of rotation to the specific case under consideration. Nevertheless, anticipating the discussion to follow in sections 4 and 5, we mention already now that the whole problem can be considered exclusively in Fourier superspace such that only the convergence properties of the characteristic function matter.

3.7. Generating function

Having obtained the probability density $Q(\sigma)$, we use equations (30) and (41) in formula (11). The remaining steps to be done are then exactly as in [12], and we arrive at the result

$$Z_k(x+J) = \int \mathrm{d}[\sigma]\, Q(\sigma) \mathrm{detg}^{-N}(\sigma - x^- J), \tag{52}$$

where

$$Q(\sigma) = \int \mathrm{d}[h]\, P^{(\mathrm{red})}(h)\, \chi(\sigma, h) \tag{53}$$

is the probability density in superspace. Although the general transformation formula (53) clearly is of conceptual interest, the reader might wonder what its practical value is. As the naive definition $P^{(\mathrm{red})}(h) = \int \mathrm{d}[H/h] P(H)$ does not yield the correct reduced probability density, a direct computation of the latter from the probability density $P(H)$ does not seem to be possible at the moment, and any calculation must involve the characteristic function. Nevertheless, as the example in appendix F shows, the naive definition yields a result which is not so far off, it differs from the right one by signs and imaginary units. Hence, there might be a way to render the naive definition correct. It is, however, much more important that, first, the correlations can be nicely expressed in terms of the reduced probability density $P^{(\mathrm{red})}(h)$ and that, second, the entire approach can be fully carried through in Fourier superspace such that everything relies exclusively on the characteristic function. These insights will be given in sections 4 and 5.

3.8. Norm-dependent ensembles revisited

The transformation formula (53) generalizes a transformation formula which we obtained for norm-dependent random matrix ensembles [10]. We revisit this case to acquire some experience with the generalized transformation formula. The probability density $P(H)$ of a norm-dependent ensemble depends on H only via $\mathrm{tr}\, H^2$. In [15], the class of these ensembles was constructed by averaging Gaussian probability densities over the variance t:

$$P(H) = \int_0^\infty f(t) \frac{1}{2^{N/2}(\pi t)^{N^2/2}} \exp\left(-\frac{1}{2t}\,\mathrm{tr}\, H^2\right) \mathrm{d}t, \tag{54}$$

where the choice of the spread function $f(t)$ determines the ensemble. As we obviously have

$$P^{(\mathrm{red})}(h) = \int_0^\infty f(t) \frac{1}{(2\pi t)^{k/2}} \exp\left(-\frac{1}{2t}\mathrm{trg}\, h^2\right) \mathrm{d}t, \tag{55}$$

we find with the transformation formula (53)

$$\begin{aligned} Q(\sigma) &= 2^{2k(k-1)} \int_0^\infty \mathrm{d}t\; f(t) \int \mathrm{d}[\rho] \exp(-\mathrm{i}\,\mathrm{trg}\,\sigma\rho) \\ &\quad \times \int \mathrm{d}[h] \frac{1}{(2\pi t)^{k/2}} \exp\left(-\frac{1}{2t}\mathrm{trg}\, h^2\right) \exp(\mathrm{i}\,\mathrm{trg}\, hr) \\ &= 2^{k(k-1)} \int_0^\infty \mathrm{d}t\; f(t) \int \mathrm{d}[\rho] \exp(-\mathrm{i}\,\mathrm{trg}\,\sigma\rho) 2^{k(k-1)} \exp\left(-\frac{t}{2}\mathrm{trg}\, r^2\right) \\ &= \int_0^\infty \mathrm{d}t\; f(t) 2^{k(k-1)} \exp\left(-\frac{1}{2t}\mathrm{trg}\,\sigma^2\right) \end{aligned} \tag{56}$$

which is indeed the correct result. We mention in passing that it allows one to express the mapping of norm-dependent ensembles from ordinary to superspace as one single integral in terms of the probability density alone [10], i.e. without explicit appearance of the spread function.

4. Formulation in Fourier superspace

Another supersymmetric formulation of the generating function will prove most helpful for calculations of the correlation functions later on. Also from a conceptual viewpoint, it has some rather appealing features. In section 4.1, we construct the new formulation by exploiting a convolution theorem, and in section 4.2 we give a direct derivation.

4.1. Applying a convolution theorem

According to equation (52), $Z_k(x+J)$ is a convolution in supermatrix space. For three $2k \times 2k$ Hermitian supermatrices σ, ρ, τ and for two well-behaved functions $g_1(\sigma)$, $g_2(\sigma)$ as well as their Fourier transforms $G_1(\rho)$, $G_2(\rho)$, one easily derives the convolution theorem

$$\int \mathrm{d}[\sigma] g_1(\sigma) g_2(\tau - \sigma) = 2^{2k(k-1)} \int \mathrm{d}[\rho] \exp(-\mathrm{i}\,\mathrm{trg}\,\tau\rho) G_1(\rho) G_2(\rho). \tag{57}$$

In the present case, we have $\tau = x + J$. We already know the Fourier transform of $Q(\sigma)$, it is just the characteristic function $\Phi(\rho)$. The Fourier transform

$$I(\rho) = \int \mathrm{d}[\sigma] \exp(\mathrm{i}\,\mathrm{trg}\,\rho\sigma)\,\mathrm{detg}^{-N}\sigma^{-}, \tag{58}$$

of the superdeterminant is needed. It can be viewed as a supersymmetric generalization of the Ingham–Siegel integral, whose ordinary version has recently been used in the framework of supersymmetric methods [17]. Obviously, $I(\rho)$ only depends on the eigenvalues r of ρ. In appendix C, we show that it is given by

$$\begin{aligned} I(\rho) &= c_{Nk} \prod_{p=1}^{k} \Theta(r_{p1})(\mathrm{i}r_{p1})^N \exp(-\varepsilon r_{p1}) \frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}} \\ c_{Nk} &= \frac{1}{2^{k(k-1)}} \left(\frac{\mathrm{i}2\pi(-1)^{N-1}}{(N-1)!} \right)^k . \end{aligned} \tag{59}$$

We note that $I(\rho)$ is almost equal to $\mathrm{detg}^{+N}\rho$, apart from the restriction to negative eigenvalues r_{p1} and the occurrence of the functions $\delta(r_{p2})$ instead of $1/r_{p2}^{\pm}$. Loosely speaking, the Fourier transform maps the superdeterminant raised to the power $-N$ onto the superdeterminant raised to the power $+N$. We find from equations (52) and (57)

$$Z_k(x+J) = 2^{2k(k-1)} \int \mathrm{d}[\rho] \exp(-\mathrm{i}\,\mathrm{trg}(x+J)\rho) \Phi(\rho) I(\rho). \tag{60}$$

Thus we arrive at the remarkable insight that only the characteristic function $\Phi(K)$ is needed in the generating function and, thus, for the calculation of the correlation functions. It is certainly interesting that the probability density in superspace $Q(\sigma)$ follows in a unique way from the one in ordinary space $P(H)$, but the use of $Q(\sigma)$ can be avoided if the Fourier superspace representation is more convenient in a particular application.

4.2. Direct derivation

Since $Q(\sigma)$ does not appear anymore in the expression (60), the question arises if one can obtain this result without going through the construction of the probability density. This is indeed possible. We go back to equation (11) and do the average over the ensemble:

$$\begin{aligned} Z_k(x+J) = \prod_{p=1}^{k} &\int \mathrm{d}[z_p] \exp\left(\mathrm{i}L_p z_p^{\dagger}(\mathrm{i}L_p\varepsilon - x_p + J_p) z_p\right) \\ &\times \int \mathrm{d}[\zeta_p] \exp\left(\mathrm{i}\zeta_p^{\dagger}(\mathrm{i}L_p\varepsilon - x_p - J_p)\zeta_p\right) \Phi(K). \end{aligned} \tag{61}$$

We now use the insights of section 3.4 and insert an integral over a δ function,

$$\begin{aligned}\Phi(K) = \Phi(B) &= \int \mathrm{d}[\rho]\Phi(\rho)\delta^{(4k^2)}(\rho - B) \\ &= 2^{2k(k-1)} \int \mathrm{d}[\rho]\Phi(\rho) \int \mathrm{d}[\sigma]\exp(-\mathrm{i}\,\mathrm{trg}\,\sigma(\rho - B)),\end{aligned} \tag{62}$$

where ρ and σ are $2k \times 2k$ Hermitian supermatrices, to which the Wick type of rotation has been applied in the fermion–fermion blocks. Again, one might argue that this makes the expressions in equation (62) ill defined, because these matrices and the matrix B are treated on equal footing, although no Wick type of rotation has been applied to the latter. The same reasoning as in section 3.4 can be employed: either one also Wick-rotates B or one argues that the integrals in equation (62) are well defined because the elements of B in the fermion–fermion block are in any case nilpotent. We plug equation (62) into equation (61). The integrals over the vectors z_p and ζ_p can then be done in the usual way, and we have

$$\begin{aligned}Z_k(x + J) &= 2^{2k(k-1)} \int \mathrm{d}[\rho]\Phi(\rho) \int \mathrm{d}[\sigma]\exp(-\mathrm{i}\,\mathrm{trg}\,\sigma\rho)\mathrm{detg}^{-N}(\sigma - x^- J) \\ &= 2^{2k(k-1)} \int \mathrm{d}[\rho]\Phi(\rho)\exp(-\mathrm{i}\,\mathrm{trg}(x + J)\rho) \int \mathrm{d}[\sigma]\exp(-\mathrm{i}\,\mathrm{trg}\,\sigma\rho)\mathrm{detg}^{-N}\sigma^+,\end{aligned} \tag{63}$$

where we shifted σ by $x + J$ in the last step. The remaining σ integral is, after changing variables from σ to $-\sigma$, precisely of the Ingham–Siegel type (58) and we obtain equation (60).

Of course, the probability density $Q(\sigma)$ is somewhat hidden in equation (62). However, to actually obtain it, one has to do the ρ integral, which would require an interchange with the σ integration. Avoiding the introduction of the probability density $Q(\sigma)$ in the derivation sheds new light on the convergence issues. If $P(H)$ is a Schwartz function, $\Phi(K)$ is a Schwartz function as well and the convergence discussion can be exclusively restricted to the Fourier superspace and to the properties of the characteristic function when passing from ordinary space, i.e. from $\Phi(K)$, to superspace, i.e. to $\Phi(B)$ and $\Phi(\rho)$.

5. Correlation functions in terms of eigenvalue integrals

In section 5.1, we briefly review the integrals that we need over the unitary group in ordinary and in superspace. We derive a first general result by identifying fundamental correlations in section 5.2. In section 5.3, we carry out the procedure outlined in section 2.3 and obtain supersymmetric integral representations of the correlation functions for arbitrary positions of the imaginary increments. Another general result is given in section 5.4, exclusively in terms of eigenvalue integrals. In section 5.5, we discuss a probability density involving higher order traces as an example.

5.1. Eigenvalue-angle coordinates and group integrals

The Hermitian random matrix is diagonalized according to $H = UEU^\dagger$ with $E = \mathrm{diag}(E_1, \ldots, E_N)$ and with U being in $SU(N)$. The volume element in these coordinates reads

$$\mathrm{d}[H] = \frac{\pi^{N(N-1)/2}}{N! \prod_{n=1}^{N-1} n!} \Delta_N^2(E)\,\mathrm{d}[E]\,\mathrm{d}\mu(U), \tag{64}$$

where we introduced the Vandermonde determinant

$$\Delta_N(E) = \det\left[E_n^{m-1}\right]_{n,m=1,\ldots,N} = \prod_{n<m}(E_n - E_m). \tag{65}$$

The invariant measure $\mathrm{d}\mu(U)$ in equation (64) is normalized to unity. We will make use of the Harish-Chandra–Itzykson–Zuber integral [18, 19]

$$\int \mathrm{d}\mu(U)\exp(\mathrm{i}\,\mathrm{tr}\,UEU^\dagger R) = \frac{\prod_{n=1}^{N-1} n!}{\mathrm{i}^{N(N-1)/2}}\frac{\det[\exp(\mathrm{i}E_nR_m)]_{n,m=1,\ldots,N}}{\Delta_N(E)\Delta_N(R)}, \tag{66}$$

where $R = \mathrm{diag}(R_1,\ldots,R_N)$ is also a diagonal matrix. In particular, we will employ this result for the case that $R_n = 0$ for $n > 2k$. This can be obtained in various ways, as for example in [20],

$$\begin{aligned}\int \mathrm{d}\mu(U)\exp(\mathrm{i}\,\mathrm{tr}\,UEU^\dagger R) = &\prod_{n=N-2k+1}^{N-1}\frac{n!}{\mathrm{i}^n}\\ &\times\frac{\det\left[\exp(\mathrm{i}E_nR_1)\cdots\exp(\mathrm{i}E_nR_{2k})1E_n\cdots E_n^{N-2k-1}\right]_{n=1,\ldots,N}}{\Delta_N(E)\Delta_{2k}(\widetilde{R})\prod_{n=1}^{2k}R_n^{N-2k}},\end{aligned} \tag{67}$$

where we write $\widetilde{R} = \mathrm{diag}(R_1,\ldots,R_{2k})$.

In superspace, the diagonalizations of the Hermitian supermatrices $\sigma = usu^\dagger$ and $\rho = vrv^\dagger$ have already been introduced in equations (33) and (34). The volume element $\mathrm{d}[\rho]$ reads in eigenvalue-angle coordinates [12]

$$\mathrm{d}[\rho] = B_k^2(r)\,\mathrm{d}[r]\,\mathrm{d}\mu(v), \tag{68}$$

where the function

$$B_k(r) = \frac{\Delta_k(r_1)\Delta_k(\mathrm{i}r_2)}{\prod_{p<q}(r_{p1}-\mathrm{i}r_{q2})} = \det\left[\frac{1}{r_{p1}-\mathrm{i}r_{q2}}\right]_{p,q=1,\ldots,k} \tag{69}$$

is the superspace equivalent of the Vandermonde determinant. The supersymmetric analogue [12, 21] of the Harish-Chandra–Itzykson–Zuber integral is given by

$$\int \mathrm{d}\mu(v)\exp(\mathrm{i}\,\mathrm{trg}\,vrv^\dagger s) = \frac{\mathrm{i}^k}{2^{k^2}\pi^k}\frac{\det[\exp(\mathrm{i}r_{p1}s_{q1})]_{p,q=1,\ldots,k}\det[\exp(\mathrm{i}r_{p2}s_{q2})]_{p,q=1,\ldots,k}}{B_k(r)B_k(s)}. \tag{70}$$

As in [13, 14], the normalization of the invariant measure $\mathrm{d}\mu(v)$ is chosen such that formula (70), when applied to a shifted Gaussian distribution, yields the proper δ function in the curved space of the eigenvalues for vanishing variance.

5.2. *General result as an average over the fundamental correlations*

The supergroup integral (70) can now directly be applied to the Fourier superspace formulation (60), because both the functions $\Phi(\rho)$ and $I(\rho)$ depend only on the eigenvalues r. This is the merit compared to the original superspace formulation (52), to which the result (70) cannot be applied in general. In the case of a Gaussian probability density, a shift of the integration matrix σ by $x + J$ gives a form suited for the application of the supergroup integral [12–14]. In the general case, however, this is not possible and equation (60) is much more convenient. We find

$$Z_k(x+J) = 1 + \frac{2^{k(k-1)}}{B_k(x+J)}\left(\frac{\mathrm{i}}{2\pi}\right)^k\int \mathrm{d}[r]B_k(r)\exp(-\mathrm{i}\,\mathrm{trg}(x+J)r)\Phi(r)I(r). \tag{71}$$

Two remarks are in order. The first term, i.e. unity, stems from a certain boundary contribution which only appears in superspace. In physics, it is often referred to as Efetov–Wegner–Parisi–Sourlas term [1, 22–24], while it goes by the name Rothstein contribution [25] in mathematics.

In the present case, it yields the normalization $Z_k(x) = 1$ of the generating function, because one easily sees that $1/B_k(x + J)$ vanishes at $J = 0$. Formally, the boundary contribution is obtained by putting $\rho = 0$ in the integral (60), by using $\Phi(0) = 1$ and $I(0) = 1/2^{k(k-1)}$ according to equation (59) and to appendix C, and by finally dividing the result with the factor $2^{k(k-1)}$ which is due to our definition of the volume element $\mathrm{d}[\rho]$. There are various methods to explicitly justify this procedure in the case $k = 1$. In [26], for example, it is directly constructed from Rothstein's theorem. However, there is a problem, because none of those explicit methods could be extended so far to our eigenvalue-angle coordinates for $k > 1$. We can thus not exclude that further boundary contributions exist. Nevertheless, as to be discussed below, we are confident that they are not important for our purposes.

The second remark concerns the determinants in the formula (70) which are not present in equation (71). As the functions $\Phi(r)$ and $I(r)$ are invariant under permutations of the variables r_{p1} as well as of the variables r_{p2}, it suffices to keep only one term of each determinant, because all others yield the same under the integral.

We can now proceed in different ways. Here, we begin with inserting the characteristic function in the form (45) as given in section 3.6. Upon interchanging the r and the h integral we find the expression

$$Z_k(x + J) = 1 + \frac{(-\pi)^k}{B_k(x + J)} \int \mathrm{d}[h] P^{(\mathrm{red})}(h) \widehat{R}_k^{(\mathrm{fund})}(x + J - h), \tag{72}$$

where we introduced the fundamental correlation function

$$\widehat{R}_k^{(\mathrm{fund})}(s) = 2^{k(k-1)} \int \mathrm{d}[r] B_k(r) \exp(-\mathrm{i}\,\mathrm{trg}\, sr) I(r) \tag{73}$$

as a new object. In equation (72), we have to set $s = x + J - h$. We refer to the correlation function (73) as fundamental, for it gives all structural information about the correlations before averaging over the reduced probability density $P^{(\mathrm{red})}(h)$. The fundamental correlation function is the Fourier transform of the function $I(r)$ in the curved eigenvalue space. It is closely related to the backtransform of $I(\rho)$, i.e. to the superdeterminant $\mathrm{detg}^{-N}\sigma^-$, but it is not quite the same. We discuss that in appendix D.

The result (72) is not a trivial reformulation of equation (4) defining the generating function. While it is obvious from equation (4) that only the N eigenvalues of H are relevant for the ensemble average, equation (72) makes a different statement, namely that only $2k$ diagonal elements of H enter the computation of the average. Using the determinant structure (69) of $B_k(r)$ and formula (59), we find that the fundamental correlation function has the determinant structure

$$\widehat{R}_k^{(\mathrm{fund})}(s) = \det[\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2})]_{p,q=1,\ldots,k}, \tag{74}$$

where the fundamental kernel is given by

$$\begin{aligned}\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) = -\frac{(-1)^{N-1}}{\pi(N-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}r_1\, \mathrm{d}r_2}{r_1 - \mathrm{i}r_2} \\ \times \exp\left(-\mathrm{i}\left(r_1 s_{p1}^+ + r_2 s_{q2}\right)\right) \Theta(r_1) (\mathrm{i}r_1)^N \frac{\partial^{N-1}\delta(r_2)}{\partial r_2^{N-1}}.\end{aligned} \tag{75}$$

We suppress the indices p and q in the integration variables r_1 and r_2. It is shown in appendix D that the fundamental kernel can be written as

$$\begin{aligned}\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) &= -\frac{1}{\pi} \sum_{n=0}^{N-1} \frac{(\mathrm{i}s_{q2})^n}{n!} \int_0^\infty \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp(-\mathrm{i}r_1 s_{p1}^-) \\ &= \frac{1}{\pi} \sum_{n=0}^{N-1} \frac{(\mathrm{i}s_{q2})^n}{(s_{p1}^-)^{n+1}}.\end{aligned} \tag{76}$$

As this is a finite geometric series, we may also write

$$\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) = \frac{1}{\pi s_{p1}^-} \frac{1-(\mathrm{i}s_{q2}/s_{p1}^-)^N}{1-(\mathrm{i}s_{q2}/s_{p1}^-)} = \frac{1}{\pi (s_{p1}^-)^N} \frac{(s_{p1}^-)^N - (\mathrm{i}s_{q2})^N}{s_{p1}^- - \mathrm{i}s_{q2}}. \tag{77}$$

The fact that the fundamental kernel has a representation as a finite series and as a ratio of differences is reminiscent of and related to the Christoffel–Darboux formula [27] in the theory of orthogonal polynomials.

The correlation functions according to equation (3) are then quickly obtained using the steps of [12]. We find

$$\widehat{R}_k(x_1, \ldots, x_k) = \int \mathrm{d}[h] P^{(\mathrm{red})}(h) \widehat{R}_k^{(\mathrm{fund})}(x-h) = \int \mathrm{d}[h] P^{(\mathrm{red})}(h) \times \det[\widehat{C}^{(\mathrm{fund})}(x_p - H_{pp}, x_q - \mathrm{i}H_{(k+q)(k+q)})]_{p,q=1,\ldots,k}. \tag{78}$$

The correlation functions are convolutions of the fundamental correlations with the reduced probability density $P^{(\mathrm{red})}(h)$. In the original problem, N^2 integrals over the variables in H have to be done. This is also the number of integrals to be done when calculating the characteristic function. Hence, depending on the complexity of $P(H)$ under consideration, it can be hard or even impossible to obtain $P^{(\mathrm{red})}(h)$ explicitly. Nevertheless, the result (78) yields a deep structural insight. It holds for arbitrary unitarily invariant random matrix ensembles.

It is somewhat surprising that the reduced probability density $P^{(\mathrm{red})}(h)$ suffices to write down equation (78). One might conclude that this obliterates the above convergence discussion related to the functional forms of the probability densities in ordinary and superspace and of the characteristic function. Formula (78) indeed gives reason to be optimistic. However, we recall that the characteristic function was used in the derivation, even though it does not appear any more explicitly. Moreover, we assume that the h and the r integrations can be interchanged when going from equation (43) to equation (72).

The inherent determinant structure (74) of the fundamental correlations will be destroyed in general when averaging over the random matrix ensemble. It will be preserved if the reduced probability density factorizes according to

$$P^{(\mathrm{red})}(h) = \prod_{p=1}^{k} P_1^{(\mathrm{red})}(H_{pp}) P_2^{(\mathrm{red})}(H_{(k+p)(k+p)}). \tag{79}$$

However, this is not the only situation in which the determinant structure survives. The Mehta–Mahoux theorem [7] implies that the correlation functions $R_k(x_1, \ldots, x_k)$ can be written as determinants for all unitarily invariant probability densities which factorize in their eigenvalue dependence,

$$P(H) = P(E) = \prod_{n=1}^{N} P^{(\mathrm{ev})}(E_n). \tag{80}$$

One would not expect that the factorizations (79) or (80) are completely independent, but we have not looked into this further. In the present context, it is more important that the applicability of the Mehta–Mahoux theorem is limited to precisely the case when the factorization (80) holds. It is thus a quite attractive feature of the result (78) that it is valid for all unitarily invariant probability densities which have the property $P(H) = P(E)$, but which do not need to have any factorization property as in equation (79) or (80). In this sense, formula (78) is more general than the Mehta–Mahoux theorem. In section 5.4 we will give another result, also valid for all unitarily invariant probability densities. Since it is formulated in terms of integrals over the eigenvalues, its structure is somewhat different from formula (78).

As an easy check of our findings, we show in appendix E that equation (78) yields immediately the GUE correlation functions. This is important, because it strengthens our confidence that we treated the Efetov–Wegner–Parisi–Sourlas term [1, 22–24] consistently.

5.3. Correlations functions of the imaginary parts and for arbitrary positions of the imaginary increments

As discussed in section 2.1, the correlation functions $R_k(x_1,\ldots,x_k)$ as defined in equation (1) are the main object of our interest. We now construct integral representations for them and, in addition, also for all correlation functions $\widehat{R}_k(x_1,\ldots,x_k)$ as defined in equation (2) for arbitrary positions of the imaginary increments. To avoid introduction of hyperbolic symmetry, we restricted ourselves from section 3.4 on to the case that all imaginary increments lie on the same side of the real axis. However, applying the strategy outlined in section 2.3, we can recover every correlation function that we want.

It is convenient to use the general result (78), allowing us to conduct the construction by only looking at the fundamental correlation function $\widehat{R}_k^{(\mathrm{fund})}(x-h)$. Due to its determinant structure, it depends on one fixed energy x_p either in the form $\widehat{C}^{(\mathrm{fund})}(x_p - H_{pp}, x_p - \mathrm{i}H_{(k+p)(k+p)})$ or in the form $\widehat{C}^{(\mathrm{fund})}(x_q - H_{qq}, x_p - \mathrm{i}H_{(k+p)(k+p)})\widehat{C}^{(\mathrm{fund})}(x_p - H_{pp}, x_{q'} - \mathrm{i}H_{(k+q')(k+q')})$ where $q \neq p$ and $q' \neq p$. From the first of expressions (76) we conclude that in both cases the dependence of $\widehat{R}_k^{(\mathrm{fund})}(x-h)$ on the fixed energy x_p is a finite sum of the terms

$$\widehat{\Lambda}_{nm}(x_p) = (x_p - \mathrm{i}H_{(k+p)(k+p)})^m \int_0^\infty \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp(-\mathrm{i}r_1(x_p^- H_{pp})), \tag{81}$$

where $n = m$ is possible. As the average over the ensemble is linear, it suffices to investigate the functions $\widehat{\Lambda}_{nm}(x_p)$ in order to study the energy dependence of the correlation functions $\widehat{R}_k(x_1,\ldots,x_k)$. According to section 2.3, we study the Fourier transform

$$\widehat{\lambda}_{nm}(t_p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}x_p \exp\left(\mathrm{i}t_p x_p\right) \widehat{\Lambda}_{nm}(x_p). \tag{82}$$

Shifting x_p by H_{pp}, it can be cast into the form

$$\begin{aligned}
\widehat{\lambda}_{nm}(t_p) &= \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}t_p H_{pp}) \int_{-\infty}^{+\infty} \mathrm{d}x_p \exp(\mathrm{i}t_p x_p) \\
&\quad \times (x_p + H_{pp} - \mathrm{i}H_{(k+p)(k+p)})^m \int_0^\infty \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp(-\mathrm{i}r_1 x_p^-) \\
&= \frac{1}{\sqrt{2\pi}} \exp(\mathrm{i}t_p H_{pp}) \left(H_{pp} - \mathrm{i}H_{(k+p)(k+p)} - \mathrm{i}\frac{\partial}{\partial t_p}\right)^m \\
&\quad \times \int_0^\infty \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp(-\varepsilon r_1) \int_{-\infty}^{+\infty} \mathrm{d}x_p \exp(\mathrm{i}x_p(t_p - r_1)) \\
&= \sqrt{2\pi} \exp(\mathrm{i}t_p H_{pp}) \left(H_{pp} - \mathrm{i}H_{(k+p)(k+p)} - \mathrm{i}\frac{\partial}{\partial t_p}\right)^m \\
&\quad \times \int_0^\infty \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp(-\varepsilon r_1)\delta(t_p - r_1).
\end{aligned} \tag{83}$$

As the r_1 integration extends over the positive real axis only, the integral is zero whenever $t_p < 0$. All derivatives are zero as well in this case, implying that the whole expression is proportional to $\Theta(t_p)$. For $t_p > 0$, the integral yields $(\mathrm{i}t_p)^n \exp(-\varepsilon t_p)$. All derivatives of the exponential function give terms containing powers of ε and thus vanish in the limit $\varepsilon \to 0$.

Here, we may assume that the t_p integral cannot yield bare singularities in ε. We can thus neglect all these terms and write $\exp(-\varepsilon t_p)$ in front of the entire expression. We find

$$\widehat{\lambda}_{nm}(t_p) = \mathrm{i}2\Theta(t_p)\exp(-\varepsilon t_p)\lambda_{nm}(t_p), \tag{84}$$

where

$$\lambda_{nm}(t_p) = \frac{\sqrt{2\pi}}{\mathrm{i}2}\exp(\mathrm{i}t_p H_{pp})\left(H_{pp} - \mathrm{i}H_{(k+p)(k+p)} - \mathrm{i}\frac{\partial}{\partial t_p}\right)^m (\mathrm{i}t_p)^n. \tag{85}$$

Indeed, equation (84) directly implies equation (7) and we can read off the desired integral representations. The function $\lambda_{nm}(t_p)$ is recognized as Fourier transform of

$$\begin{aligned}\Lambda_{nm}(x_p) &= -\mathrm{i}(x_p - \mathrm{i}H_{(k+p)(k+p)})^m \int_{-\infty}^{+\infty} \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp(-\mathrm{i}r_1(x_p - H_{pp})),\\ &= -\mathrm{i}(-1)^n 2\pi (x_p - \mathrm{i}H_{(k+p)(k+p)})^m \frac{\partial^n}{\partial x_p^n}\delta(x_p - H_{pp})\\ &= -2n!(x_p - \mathrm{i}H_{(k+p)(k+p)})^m \operatorname{Im}\frac{1}{(x_p^- - H_{pp})^{n+1}}.\end{aligned} \tag{86}$$

Collecting everything, we arrive at

$$R_k(x_1, \ldots, x_k) = \int \mathrm{d}[h] P^{(\mathrm{red})}(h) R_k^{(\mathrm{fund})}(x - h) \tag{87}$$

with the fundamental correlation function

$$R_k^{(\mathrm{fund})}(s) = \det[C^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2})]_{p,q=1,\ldots,k} \tag{88}$$

and the fundamental kernel

$$\begin{aligned}C^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) &= -\frac{1}{2\pi}\sum_{n=0}^{N-1}\frac{(\mathrm{i}s_{q2})^n}{n!}\int_{-\infty}^{+\infty}\mathrm{d}r_1(\mathrm{i}r_1)^n\exp(-\mathrm{i}r_1 s_{p1})\\ &= \frac{1}{\pi}\sum_{n=0}^{N-1}(\mathrm{i}s_{q2})^n \operatorname{Im}\frac{1}{(s_{p1}^-)^{n+1}}.\end{aligned} \tag{89}$$

Hence one simply has to replace the singularities $1/(s_{p1}^-)^{n+1}$ everywhere with their imaginary parts. Tracing back these considerations, we realize that all necessary modifications reside in the r_{p1} integrals and specifically in the function $I(r)$. Replacing equation (59) with

$$I(\rho) = \frac{1}{2^{k(k-1)}}\left(\frac{\pi(-1)^{N-1}}{(N-1)!}\right)^k \prod_{p=1}^{k}(\mathrm{i}r_{p1})^N \frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}} \tag{90}$$

is equivalent to the above-discussed steps made to obtain $R_k(x_1, \ldots, x_k)$.

With the help of formula (10), it is now an easy exercise to construct integral representations for the correlation functions $\widehat{R}_k(x_1, \ldots, x_k)$ defined in equation (2) with arbitrary positions of the imaginary increments. Formulae (74) and (78) remain valid if the fundamental kernel is replaced with

$$\begin{aligned}\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) &= \mp\frac{1}{\pi}\sum_{n=0}^{N-1}\frac{(\mathrm{i}s_{q2})^n}{n!}\int_0^{\infty}\mathrm{d}r_1(\mathrm{i}r_1)^n\exp\left(\mp\mathrm{i}r_1 s_{p1}^{\mp}\right)\\ &= \frac{1}{\pi}\sum_{n=0}^{N-1}\frac{(\mathrm{i}s_{q2})^n}{\left(s_{p1}^{\mp}\right)^{n+1}} = \frac{1}{\pi\left(s_{p1}^{\mp}\right)^N}\frac{\left(s_{p1}^{\mp}\right)^N - (\mathrm{i}s_{q2})^N}{s_{p1}^{\mp} - \mathrm{i}s_{q2}},\end{aligned} \tag{91}$$

where the notation $s_{p1}^{\mp}$ indicates that the imaginary increment is chosen according to $x_p^{\pm} = x_p - \mathrm{i}L_p\varepsilon$. In terms of the function $I(r)$, this is equivalent to replacing equation (59) with

$$I(\rho) = c_{Nk} \prod_{p=1}^{k} \Theta(L_p r_{p1})(\mathrm{i}r_{p1})^N \exp(-L_p \varepsilon r_{p1}) \frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}}. \tag{92}$$

Thus, we obtain supersymmetric integral representations for all these correlation functions without using hyperbolic symmetry.

The insights just presented may be viewed as a more formal justification of the procedure denoted by the operator symbol $\Im$ in [12–14]. We argued in these studies that the generating functions satisfy a diffusion process. The diffusion propagator contains no information about the positions of the imaginary increments, this is exclusively contained in the initial condition of the diffusion. Moreover, the diffusion propagator is nothing but the supersymmetric Harish-Chandra–Itzykson–Zuber integral (70) over the unitary supergroup, not involving any non-compact degrees of freedom. This can be verified in an elementary way by simply plugging it into the diffusion equation. Hence, one is free to adjust the positions of the imaginary increments as needed, which essentially defined the operator $\Im$. We have now given another justification. Nevertheless, it remains an interesting mathematical question to also derive exact, non-asymptotic supersymmetric integral representations for arbitrary choices of the metric L from group integrals involving non-compact degrees of freedom.

5.4. General result in terms of eigenvalue integrals

A further integral representation follows from equation (71). We take the derivatives with respect to the source variables as in [12] and in section 5.2 and find

$$\widehat{R}_k(x_1, \ldots, x_k) = 2^{k(k-1)} \int \mathrm{d}[r] B_k(r) \exp(-\mathrm{i}\,\mathrm{trg}\, xr) \Phi(r) I(r). \tag{93}$$

The correlation functions $R_k(x_1, \ldots, x_k)$ as well as those for arbitrary positions of the imaginary increments are obtained as in section 5.3, we simply have to replace $I(r)$ according to equation (59) by $I(r)$ according to equation (90) or (92), respectively. We expand the determinant $B_k(r)$ by introducing the permutations ω of the indices $p = 1, \ldots, k$ and write

$$\begin{aligned}\widehat{R}_k(x_1, \ldots, x_k) = 2^{k(k-1)} c_{Nk} \sum_{\omega} (-1)^{j(\omega)} \int \mathrm{d}[r] \Phi(r) \prod_{p=1}^{k} \frac{\exp(-\mathrm{i}x_p r_{p1} - x_{\omega(p)} r_{\omega(p)2})}{r_{p1} - \mathrm{i}r_{\omega(p)2}} \\ \times \Theta(L_p r_{p1})(\mathrm{i}r_{p1})^N \exp(-L_p \varepsilon r_{p1}) \frac{\partial^{N-1}\delta(r_{\omega(p)2})}{\partial r_{\omega(p)2}^{N-1}},\end{aligned} \tag{94}$$

where $j(\omega)$ is the parity of the permutation ω. The δ functions allow us to do the k integrals over the variables r_{p2} immediately. We integrate by parts and use Leibnitz' rule to work out the derivatives of products:

$$\begin{aligned}\widehat{R}_k(x_1, \ldots, x_k) = (\mathrm{i}2\pi)^k \sum_{\omega} (-1)^{j(\omega)} \sum_{n_1=0}^{N-1} \frac{1}{n_1!} \int_{-\infty}^{+\infty} \mathrm{d}r_{11} \Theta(L_1 r_{11}) \exp(-\mathrm{i}x_1 r_{11} - L_1 \varepsilon r_{11}) \\ \times (-\mathrm{i}r_{11})^{n_1} \cdots \sum_{n_k=0}^{N-1} \frac{1}{n_k!} \int_{-\infty}^{+\infty} \mathrm{d}r_{k1} \Theta(L_k r_{k1}) \exp(-\mathrm{i}x_k r_{k1} - L_k \varepsilon r_{k1})(-\mathrm{i}r_{k1})^{n_k} \\ \times \left. \frac{\partial^{\sum_{p=1}^{k} n_p} \exp\left(-\sum_{p=1}^{k} x_{\omega(p)} r_{p2}\right) \Phi(r)}{\prod_{p=1}^{k} \partial r_{p2}^{n_p}} \right|_{r_2=0}.\end{aligned} \tag{95}$$

This result is valid for an arbitrary unitarily invariant probability density. The structure of this expression is quite different from the one in section 5.2, where the correlation functions were found to be a convolution of the reduced probability density with the fundamental correlations.

It is instructive to see how the correlation functions can acquire a determinant structure. An obvious feature leading to this would be a factorization

$$\Phi(r) = \prod_{p=1}^{k} \Phi_1^{(\mathrm{ev})}(r_{p1})\Phi_2^{(\mathrm{ev})}(r_{p2}) \tag{96}$$

of the characteristic function. We find immediately

$$\widehat{R}_k(x_1, \ldots, x_k) = \det[\widehat{C}(x_p, x_q)]_{p,q=1,\ldots,k} \tag{97}$$

with the kernel

$$\begin{aligned}\widehat{C}(x_p, x_q) = \frac{\mathrm{i}}{\pi} \sum_{n=0}^{N-1} \frac{1}{n!} \frac{\partial^n \exp(-x_q r_2)\Phi_2^{(\mathrm{ev})}(r_2)}{\partial r_2^n}\bigg|_{r_2=0} \\ \times \int_{-\infty}^{+\infty} \mathrm{d}r_1 \Theta(L_p r_1) \exp(-\mathrm{i}x_p r_1 - L_p \varepsilon r_1)(-\mathrm{i}r_1)^n \Phi_1^{(\mathrm{ev})}(r_1),\end{aligned} \tag{98}$$

where we suppress the indices p and q in the r variables. It is nicely consistent with the discussion in section 5.2 that the functions $\Phi_1^{(\mathrm{ev})}(r_{p1})$ and $\Phi_2^{(\mathrm{ev})}(r_{p2})$ are, according to equation (45), the Fourier transforms of the functions $P_1^{(\mathrm{red})}(H_{pp})$ and $P_2^{(\mathrm{red})}(H_{(k+p)(k+p)})$ in equation (79). Thus, the factorizations (79) and (96) are equivalent. We note that the GUE case is trivially recovered. We then have $\Phi_j^{(\mathrm{ev})}(r_{pj}) = \exp\left(-r_{pj}^2/4\right)$ which combines in the derivative expression with the exponential to the generating function of the Hermite polynomials, and the integral yields the generalized Hermite functions as given in appendix E. It is conceivable that mechanisms other than following from the factorizations (79) or (96) can be identified that also lead to a determinant structure. However, as the merit of equation (95) is its completely general character and its independence of such factorizations and determinant structures, we have not explored this in Fourier superspace either.

One can wonder whether it is helpful to integrate over the group $SU(N)$, i.e. over the ordinary unitary matrix U diagonalizing H, before inserting the characteristic function $\Phi(r)$ in formula (95). With the help of equation (67) we find

$$\begin{aligned}\Phi(r) &= \frac{\pi^{N(N-1)/2}}{N! \prod_{n=1}^{N-1} n!} \int \mathrm{d}[E]\Delta_N^2(E)P(E) \int \mathrm{d}\mu(U) \exp(\mathrm{i}\,\mathrm{tr}\, UEU^\dagger R) \\ &= \frac{\pi^{N(N-1)/2}}{\mathrm{i}^{(N-k)(2k-1)} N! \prod_{n=1}^{N-2k} n! \Delta_{2k}(r_1, r_2) \prod_{p=1}^{2k} (r_{p1} r_{q2})^{N-2k}} \int \mathrm{d}[E]\Delta_N(E)P(E) \\ &\quad \times \det\left[\exp(\mathrm{i}E_n r_{11}) \cdots \exp(\mathrm{i}E_n r_{k2}) 1 E_n \cdots E_n^{N-2k-1}\right]_{n=1,\ldots,N},\end{aligned} \tag{99}$$

where we have to set $R_p = r_{p1}$, $R_{p+k} = r_{p2}$, $p = 1, \ldots, k$. The eigenvalues r_{p2} do not come with imaginary units in the formula above. This is also important in $\Delta_{2k}(r_1, r_2)$ which is the ordinary Vandermonde determinant of the $2k$ variables r_1 and r_2. As the whole integrand is invariant under permutations of the eigenvalues E_n, we may replace the determinant stemming from the group integration by the product of its diagonal elements, because all other terms yield the same result. However, from the resulting expression

$$\begin{aligned}\Phi(r) = \frac{\pi^{N(N-1)/2}}{\mathrm{i}^{(N-k)(2k-1)} \prod_{n=1}^{N-2k} n! \Delta_{2k}(r_1, r_2) \prod_{p=1}^{2k} (r_{p1} r_{p2})^{N-2k}} \int \mathrm{d}[E]\Delta_N(E)P(E) \\ \times \prod_{p=1}^{k} \exp(\mathrm{i}(E_p r_{p1} + E_{k+p} r_{p2})) \prod_{n=2k+1}^{N} E_n^{n-2k-1},\end{aligned} \tag{100}$$

it is not immediately obvious anymore that its limit of vanishing r_{pj} remains finite, given by the normalization $\Phi(0) = 1$.

We give the expressions (99) and (100) mainly for the sake of completeness, because they are not particularly useful in their general form. Although the powers in the denominator are not real singularities in equations (99) and (100), they become truly singular, if one tries to exchange the order of integrations and to do the r integrations first in equation (93).

5.5. Ensembles involving higher order traces as an example

As it might be helpful to illustrate our findings by an example, we consider the probability density

$$P(H) = b_{M_1 M_2} (\mathrm{tr}\, H^{M_1})^{M_2} \exp(-\mathrm{tr}\, H^2) \tag{101}$$

for a fixed pair of integers $M_1, M_2 = 0, 1, 2, \ldots$. The constant $b_{M_1 M_2}$ ensures normalization. The Gaussian case is recovered for $M_1 = 0$ or $M_2 = 0$. A few obviously meaningless cases have to be excluded, such as the choice $M_1 = M_2 = 1$, which makes the normalization integral vanish. While the probability density (101) is still in the norm-dependent class discussed in [10] for $M_1 = 2$, it is not for other values of M_1. Importantly, the probability density (101) does not factorize according to equation (79) or (80). In particular, this means that this random matrix ensemble is not covered by the Mehta–Mahoux theorem, although we do not exclude that is possible with some efforts to extend the latter properly. Formula (78) provides a direct way to calculate the correlation functions for such ensembles. However, as we aim at addressing the conceptual issues in the present contribution, we refrain from presenting the quite cumbersome expressions too explicitly. We rather sketch the calculation briefly and infer what kind of structure the correlation functions will acquire. The reduced probability density must have the form

$$P^{(\mathrm{red})}(h) = \exp(-\mathrm{trg}\, h^2) \sum_{\{m\}} a_{\{m\}} S_{\{m\}}(h), \tag{102}$$

where $a_{\{m\}}$ are constants and where

$$S_{\{m\}}(h) = \sum_{\omega} s_{\{m\}\omega} \prod_{p=1}^{2k} H_{pp}^{m_{\omega(p)}} \tag{103}$$

are certain symmetric functions, i.e. linear combinations of products involving a set $\{m\}$ of integer exponents m_p, symmetrized by summing over all permutations ω of the indices $p = 1, \ldots, 2k$. The exponents m_p are between zero and $M_1 + M_2$ with the restriction that their sum does not exceed $M_1 + M_2$. In the commonly appearing symmetric functions, the numbers $s_{\{m\}\omega}$ are unity. Here, they stand for signs and imaginary units whose origin becomes clear when realizing that the functions $S_{\{m\}}(h)$ must be expressible as linear combinations of the invariants $(\mathrm{trg}\, h^l)^{l'}$. Thus, the correlation functions are given by

$$\widehat{R}_k(x_1, \ldots, x_k) = \sum_{\{m\}} a_{\{m\}} \int \mathrm{d}[h] \exp(-\mathrm{trg}\, h^2) S_{\{m\}}(h) \widehat{R}_k^{(\mathrm{fund})}(x - h). \tag{104}$$

Upon inserting equation (103) and using the determinant structure of the fundamental correlations, we obtain

$$\widehat{R}_k(x_1, \ldots, x_k) = \sum_{\{m\}} a_{\{m\}} \sum_{\omega} s_{\{m\}\omega} \det \left[\widehat{C}_{m_{\omega(p)} m_{\omega(k+q)}}(x_p, x_q) \right]_{p,q=1,\ldots,k} \tag{105}$$

where the kernel

$$\widehat{C}_{m_1 m_2}(x_p, x_q) = \frac{1}{\pi} \exp\left(-x_p^2\right) \sum_{n=0}^{N-1} \frac{1}{n!} \widehat{\eta}_{n m_1}(x_p) \vartheta_{n m_2}(x_q) \tag{106}$$

has a structure formally similar to that of the GUE kernel. The functions

$$\begin{aligned}\widehat{\eta}_{nm_1}(x_p) &= \int_{-\infty}^{+\infty} \mathrm{d}H_{pp} \exp\left(-H_{pp}^2\right) H_{pp}^{m_1} \int_0^{\infty} \mathrm{d}r_1 (\mathrm{i}r_1)^n \exp\left(\mp \mathrm{i}r_1\left(x_p^{\mp} - H_{pp}\right)\right),\\ \vartheta_{nm_2}(x_q) &= \int_{-\infty}^{+\infty} \mathrm{d}H_{(k+q)(k+q)} \exp\left(-H_{(k+q)(k+q)}^2\right) H_{(k+q)(k+q)}^{m_2} (x_q - \mathrm{i}H_{(k+q)(k+q)})^n\end{aligned} \tag{107}$$

can be written as finite weighted sums of the generalized Hermite functions which are discussed in appendix E and of the ordinary Hermite polynomials, respectively. According to the result (105), the correlation functions are linear combinations of determinants.

Alternatively, this calculation can be carried out using the results of section 5.4. It follows from the inverse of formula (46) that the characteristic function has a form very similar to the reduced probability density,

$$\Phi(r) = \exp\left(-\frac{1}{4}\mathrm{trg}\, r^2\right) \sum_{\{m\}} \widetilde{a}_{\{m\}} S_{\{m\}}(r), \tag{108}$$

with new constants $\widetilde{a}_{\{m\}}$. With the help of equation (93) or (95) this leads in a straightforward manner to the above-mentioned linear combinations of determinants.

6. Summary and conclusions

We derived supersymmetric formulations for arbitrary unitarily invariant random matrix ensembles. The construction is based on an algebraic duality between ordinary and superspace which made it possible to generalize the Hubbard–Stratonovich transformation. We identified a reduced probability density and a projector which yield the probability density in superspace from the one in ordinary space. However, we showed that despite the conceptual insights thereby obtained, the theory can be formulated without using the probability density in superspace. It turned out that it is possible and often even better to work in Fourier space. Remarkably, the characteristic functions of the probability density have the same functional form in ordinary and superspace, if they only depend on traces involving integer powers of the matrices. At present, it appears to us that it is *a priori* easier to analyse some convergence issues in Fourier superspace, but to make more definite statements will require additional work. It is not inconceivable that manifestly invariant theories can be constructed in Fourier superspace.

This leads us to the symmetry issue. There seems to be no way around hyperbolic symmetry if one wishes to set up nonlinear σ models. Here, however, we were interested in exact, non-asymptotic results. Although this requires the calculation of certain group integrals, it simplifies the symmetry: we showed that compact supergroups are sufficient to construct supersymmetric integral representations of the correlation functions for arbitrary positions of the imaginary increments. This is a more formal justification of a procedure which we have been using in previous work. We conclude that hyperbolic symmetry is a necessity for nonlinear σ models, but not for supersymmetric theories in general. Nevertheless, even though mathematics can be nicer than one expects, it is an interesting challenge to also derive those supersymmetric integral representations from a version of the theory in terms of non-compact groups.

We gave two general results for the correlation functions. The first one involves certain correlations to which we refer as fundamental, while the second one is only in terms of eigenvalue integrals. Both results are valid for arbitrary unitarily invariant random matrix ensembles. No factorization property of the probability density has to be assumed.

Worthwhile extensions of the study presented here are the exact calculation of correlation functions for various unitarily invariant ensembles and the investigation of scaling limits and large-N behaviour—a task for which our results are particularly well suited.

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Appendix A. Equality of the traces

The assertion (22) is obviously correct for $m = 1$, because we have

$$\begin{aligned}\operatorname{tr} K &= \sum_{p=1}^{k} \operatorname{tr}\left(L_p z_p z_p^\dagger - \zeta_p \zeta_p^\dagger\right) = \sum_{p=1}^{k} \left(L_p z_p^\dagger z_p + \zeta_p^\dagger \zeta_p\right) \\ &= \operatorname{trg} B. \end{aligned} \tag{A.1}$$

For $m = 2, 3, \ldots$, we find

$$\begin{aligned}\operatorname{tr} K^m &= \operatorname{tr} ALA^\dagger \cdots ALA^\dagger \\ &= \operatorname{tr} AL^{1/2}L^{1/2}A^\dagger AL^{1/2} \cdots L^{1/2}A^\dagger AL^{1/2}L^{1/2}A^\dagger \\ &= \operatorname{tr} AL^{1/2}B^{m-1}L^{1/2}A^\dagger. \end{aligned} \tag{A.2}$$

Without anticommuting variables, we could now simply use the invariance of the trace under cyclic permutation and would arrive at the desired result (22), but with an ordinary trace also on the right-hand side. To carefully account for the anticommuting variables, we write $C = B^{m-1}$ and introduce the upper indices (c1) and (c2) in boson–fermion block notation for the commuting variables as well as (a12) and (a21) for the anticommuting ones. We obtain

$$\begin{aligned}\operatorname{tr} K^m &= \sum_{p,q} \operatorname{tr}\left(z_p L_p^{1/2} C_{pq}^{(\mathrm{c1})} L_q^{1/2} z_q^\dagger - z_p L_p^{1/2} C_{pq}^{(\mathrm{a12})} \zeta_q^\dagger + \zeta_p C_{pq}^{(\mathrm{a21})} L_q^{1/2} z_q^\dagger - \zeta_p C_{pq}^{(\mathrm{c2})} \zeta_q^\dagger\right) \\ &= \sum_{p,q} \left(L_q^{1/2} z_q^\dagger z_p L_p^{1/2} C_{pq}^{(\mathrm{c1})} + \zeta_q^\dagger z_p L_p^{1/2} C_{pq}^{(\mathrm{a12})} + L_q^{1/2} z_q^\dagger \zeta_p C_{pq}^{(\mathrm{a21})} + \zeta_q^\dagger \zeta_p C_{pq}^{(\mathrm{c2})}\right) \\ &= \sum_{p,q} \left(B_{pq}^{(\mathrm{c1})} C_{pq}^{(\mathrm{c1})} + B_{pq}^{(\mathrm{a12})} C_{pq}^{(\mathrm{a21})} - \left(B_{pq}^{(\mathrm{a21})} C_{pq}^{(\mathrm{a12})} + B_{pq}^{(\mathrm{c2})} C_{pq}^{(\mathrm{c2})}\right)\right) \\ &= \operatorname{trg} BC = \operatorname{trg} BB^{m-1} = \operatorname{trg} B^m, \end{aligned} \tag{A.3}$$

as claimed.

Appendix B. Details of the spectral decomposition

The matrix K is ordinary Hermitian, although anticommuting variables are present. In particular, all inverses of the matrix elements $K_{n'n}$ exist. The eigenvalues $\widetilde{Y}_n$ are thus uniquely defined. Moreover, the diagonalizing matrix $\widetilde{V} = [\widetilde{V}_1 \cdots \widetilde{V}_N]$ introduced in equation (15) is ordinary unitary and in $SU(N)$, and the corresponding eigenvectors $\widetilde{V}_n$ are orthonormal and have commuting elements only. This might seem to be at odds with the form (25) of the eigenvectors V_n. In the second representation of the supervectors w_n, the w_{np1} are anticommuting and the w_{np2} are commuting, such that all elements of the vector V_n are anticommuting, despite the fact that K is an ordinary matrix. To clarify this, we note that the

eigenvectors V_n can be written in the form $V_n = \gamma_n \widehat{V}_n$ where the coefficients γ_n are commuting in the first and anticommuting in the second representation of the vectors w_n. The V_n are orthogonal, but they cannot be normalized in the standard way, if the γ_n are anticommuting. We emphasize that this causes no problem whatsoever. All entries of the vectors $\widehat{V}_n$ are commuting, although they contain anticommuting variables. The eigenvalue equation (23) thus reads $K\gamma_n\widehat{V}_n = Y_n\gamma_n\widehat{V}_n$. Since the possibly anticommuting structure due to γ_n does not interfere with the matrix structure of K, the vectors $\widehat{V}_n$ must be eigenvectors. As the eigenvectors $\widetilde{V}_n$ for the non-zero eigenvalues of K are unique, we may identify them in these cases with the vectors $\widehat{V}_n$. The eigenvectors for the zero eigenvalues are not important in our context and do not need to be specified. Hence, we have

$$V_n = \gamma_n \widetilde{V}_n, \tag{B.1}$$

such that the eigenvectors V_n and $\widetilde{V}_n$ are proportional to each other. The coefficients can be written as the scalar products

$$\gamma_n = \widetilde{V}_n^\dagger V_n = \widetilde{V}_n^\dagger A L^{1/2} w_n. \tag{B.2}$$

If γ_n is anticommuting, it is has to be nilpotent, which means that an integer j exists such that $\gamma_n^j = 0$. We note that γ_n is not nilquadratic, i.e. the number j is here larger than two, because γ_n is a complicated linear combination of nilquadratic anticommuting variables. In general, a linear combination of J nilquadratic anticommuting variables is nilpotent for every $j > J+1$. Moreover, we also deduce from equation (B.1) that

$$V_{n'}^\dagger V_n = \gamma_n^* \gamma_n \delta_{n'n} \tag{B.3}$$

is the orthogonality relation.

It is worthwhile to also collect more information about the supermatrix B. According to the definition (20), it is non-Hermitian and satisfies

$$B^\dagger = LBL. \tag{B.4}$$

One easily sees that

$$w_{p'}^\dagger L w_p = \delta_{p'p} \tag{B.5}$$

is the corresponding orthonormality relation for the eigenvectors w_p. Being supervectors, these eigenvectors can always be properly normalized to unity. One has

$$\sum_{p=1}^{2k} w_p w_p^\dagger L = 1_{2k} \tag{B.6}$$

as the completeness relation.

We construct a helpful alternative representation of the matrix K. Employing the form (25) and the completeness relation (B.6) we work out the expression

$$\begin{aligned}\sum_{n=1}^{2k} V_n V_n^\dagger &= \sum_{n=1}^{2k} A L^{1/2} w_n w_n^\dagger (L^{1/2})^\dagger A^\dagger \\ &= \sum_{n=1}^{2k} A L^{1/2} w_n w_n^\dagger L L^{1/2} A^\dagger = ALA^\dagger,\end{aligned} \tag{B.7}$$

and by virtue of equation (19) we arrive at

$$K = \sum_{n=1}^{2k} V_n V_n^\dagger. \tag{B.8}$$

This spectral decomposition is somewhat strange, because the eigenvalues do not appear explicitly. However, useful results can be deduced from it. In the eigenvalue equation $K V_n = Y_n V_n$ it gives together with the orthogonality relation (B.3)

$$Y_n = V_n^\dagger V_n = \gamma_n^* \gamma_n. \tag{B.9}$$

Thus, the k eigenvalues $Y_{p+k} = y_{p2}$ are products of two nilpotent anticommuting variables. From the decomposition (B.8), we may also conclude

$$K \widetilde{V}_n = \sum_{n'=1}^{2k} V_{n'} V_{n'}^\dagger \widetilde{V}_n = V_n V_n^\dagger \widetilde{V}_n = \gamma_n \widetilde{V}_n \gamma_n^* = \gamma_n \gamma_n^* \widetilde{V}_n, \tag{B.10}$$

where we used equations (B.1) and (B.2). In the case of the second representation, in which γ_n is anticommuting, comparison with the definition (15) yields

$$\widetilde{Y}_n \widetilde{V}_n = K \widetilde{V}_n = -\gamma_n^* \gamma_n \widetilde{V}_n, \tag{B.11}$$

implying together with equation (B.9) the sign switch $\widetilde{Y}_n = -Y_n$ for all eigenvalues corresponding to the second representation. This reasoning is consistent with an alternative derivation based on matrix invariants which is given by Wei [16].

Furthermore, one readily sees from the decomposition (B.8) that all eigenvalues $Y_n, n > 2k$ which are different from $Y_p = y_{p1}$ and $Y_{p+k} = y_{p2}$ must be zero. As K is Hermitian, one can convince oneself in the usual way that the eigenvectors V_n to different eigenvalues are orthogonal. Let V_n be an eigenvector to an eigenvalue Y_n with $n > 2k$. We immediately conclude from equation (B.8) that $K V_n = 0$ and hence $Y_n = 0$.

We show in an alternative way that the $2k$ non-zero eigenvalues Y_n of K coincide with the eigenvalues of B. We write the eigenvalue equation as $B w_n = b_n w_n$ and consider the orthogonality relation

$$\begin{aligned} \delta_{n'n} Y_n &= V_{n'}^\dagger V_n \\ &= w_{n'}^\dagger (L^{1/2})^\dagger A^\dagger A L^{1/2} w_n = w_{n'}^\dagger (L^{1/2})^\dagger L^{-1/2} L^{1/2} A^\dagger A L^{1/2} w_n \\ &= w_{n'}^\dagger L B w_n = b_n w_{n'}^\dagger L w_n = b_n \delta_{n'n}. \end{aligned} \tag{B.12}$$

For $n = n'$ we conclude $b_n = Y_n$ as claimed, and for $n \neq n'$ we observe that the orthogonalities of the eigenvectors V_n and w_n mutually imply each other.

Finally, we mention that the relation $\mathrm{tr}\, K^m = \mathrm{tr}\, \widetilde{Y}^m$ requires a careful interpretation for non-integer m. This is so, because the expression $(\gamma_n^* \gamma_n)^m$ is ill defined for non-integer m if γ_n is anticommuting. However, one can avoid the diagonalization altogether and decompose $K = K^{(\mathrm{c})} + K^{(\mathrm{a})}$ where $K^{(\mathrm{c})}$ and $K^{(\mathrm{a})}$ contain the dyadic matrices with commuting and anticommuting variables, respectively. Thereby the issue is mapped onto the question whether the infinite power series $\mathrm{tr}\, K^m = \mathrm{tr}(K^{(\mathrm{c})} + K^{(\mathrm{a})})^m$ is well defined for non-integer m. This seems doubtful, because the $N \times N$ matrix $K^{(\mathrm{c})}$ has rank k with $k < N$.

Appendix C. A supersymmetric Ingham–Siegel integral

For $2k \times 2k$ Hermitian supermatrices σ and ρ, we wish to calculate the integral $I(\rho)$, i.e. the Fourier transform (58) of the superdeterminant. As $I(\rho)$ is obviously an invariant function depending on eigenvalues only, we may replace ρ with r. Up to a certain point, we can apply and slightly extend the methods given in [17] for the case of ordinary matrices.

Employing the notation of appendix A, the matrix σ is viewed as consisting of the element $\sigma_{11}^{(c1)}$, the supervector $\vec{\sigma}_1 = \left(\sigma_{21}^{(c1)}, \ldots, \sigma_{k1}^{(c1)}, \sigma_{11}^{(a21)}, \ldots, \sigma_{k1}^{(a21)}\right)$ with $k-1$ commuting and k anticommuting variables, the complex conjugate $\vec{\sigma}_1^\dagger$ and the $(2k-1)\times(2k-1)$ Hermitian supermatrices $\widetilde{\sigma}$ containing all other matrix elements. Because of

$$\mathrm{detg}\sigma^- = \mathrm{detg}\widetilde{\sigma}^- \left(\sigma_{11}^{(c1)-} + \vec{\sigma}_1^\dagger (\widetilde{\sigma}^-)^{-1}\vec{\sigma}_1\right) \tag{C.1}$$

the integral over $\sigma_{11}^{(c1)}$ can easily be performed with the help of the residue theorem. Some care is needed, because the bilinear form $\vec{\sigma}_1^\dagger(\widetilde{\sigma}^-)^{-1}\vec{\sigma}_1$ is an undetermined complex number due to the presence of the imaginary increments. However, as the variables $\widetilde{\sigma}$ are only parameters in the $\sigma_{11}^{(c1)}$ integration, we may shift the imaginary increments away, assuming that $\widetilde{\sigma}$ can be inverted. The unitary supermatrix diagonalizing $\widetilde{\sigma}$ can then be absorbed into the supervector $\vec{\sigma}_1$. This makes the bilinear form $\vec{\sigma}_1^\dagger(\widetilde{\sigma})^{-1}\vec{\sigma}_1$ real, and the residue is well determined. The integral over the supervector $\vec{\sigma}_1$ is then simply Gaussian and we find

$$I(\rho) \sim \Theta(r_{11})(\mathrm{i}r_{11})^N \exp(-\varepsilon r_{11}) \int \mathrm{d}[\widetilde{\sigma}] \exp(\mathrm{i}\,\mathrm{trg}\,\widetilde{r}\widetilde{\sigma})\,\mathrm{detg}^{-(N-1)}\widetilde{\sigma}^- \tag{C.2}$$

with $\widetilde{r} = \mathrm{diag}(r_{21}, \ldots, r_{k1}, \mathrm{i}r_{k2}, \ldots, \mathrm{i}r_{k2})$. We perform the calculation up to the normalization constant which will be determined later on. It should be noted that the presence of the anticommuting variables leads to some differences as compared to the corresponding formula in [17]. We can repeat this step $k-1$ further times until all variables $\sigma_{pq}^{(c1)}$ and all anticommuting variables $\sigma_{pq}^{(a21)}$ and $\sigma_{pq}^{(a12)}$ have been integrated out. This results in

$$\begin{aligned} I(\rho) &\sim \prod_{p=1}^{k} \Theta(r_{p1})(\mathrm{i}r_{p1})^N \exp(-\varepsilon r_{p1}) J(r_2) \\ J(r_2) &= \int \mathrm{d}[\sigma^{(c2)}] \exp(\mathrm{i}\,\mathrm{tr}\, r_2\sigma^{(c2)})\mathrm{det}^{N-k}\sigma^{(c2)}. \end{aligned} \tag{C.3}$$

The remaining integral $J(r_2)$ is over the ordinary $k\times k$ Hermitian matrix $\sigma^{(c2)}$. As no anticommuting variables appear in the integrand, the inverse superdeterminant is identical to the determinant in the numerator. This determinant does not contain singularities anymore and thus we dropped the imaginary increments. Upon introducing eigenvalue-angle coordinates for $\sigma^{(c2)}$ and applying the Harish-Chandra–Itzykson–Zuber integral (66) for $U(k)$, we are left with an integral over the eigenvalues $s_p^{(c2)}$, $p = 1, \ldots, k$ given by

$$J(r_2) \sim \frac{1}{\Delta_k(r_2)} \int \mathrm{d}[s^{(c2)}]\Delta_k(s^{(c2)}) \exp(\mathrm{i}\,\mathrm{tr}\, r_2 s^{(c2)})\,\mathrm{det}^{N-k}s^{(c2)}. \tag{C.4}$$

As the Vandermonde determinant appearing in the remaining integral reads

$$\Delta_k(s^{(c2)}) = \prod_{p<q}\left(s_p^{(c2)} - s_q^{(c2)}\right) = \det\left[\left(s_p^{(c2)}\right)^{q-1}\right]_{p,q=1,\ldots,k}, \tag{C.5}$$

we can do the eigenvalue integrals and have

$$J(r_2) \sim \frac{1}{\Delta_k(r_2)} \det\left[\frac{\partial^{N-k+q-1}\delta(r_{p2})}{\partial r_{p2}^{N-k+q-1}}\right]_{p,q=1,\ldots,k}. \tag{C.6}$$

The Vandermonde determinant $\Delta_k(r_2)$ cancels out. To see this one may use the identity

$$\frac{\partial^{N-k+q-1}\delta(r_{p2})}{\partial r_{p2}^{N-k+q-1}} = (-1)^{N-1}\frac{(N-k+q-1)!}{(N-1)!} r_{p2}^{k-q} \frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}}, \tag{C.7}$$

which is easily derived by multiplying the right-hand side with a well-behaved, non-singular function and integrating by parts. Thus, we have

$$\det\left[\frac{\partial^{N-k+q-1}\delta(r_{p2})}{\partial r_{p2}^{N-k+q-1}}\right]_{p,q=1,\ldots,k} \sim \det\left[r_{p2}^{k-q}\frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}}\right]_{p,q=1,\ldots,k}$$
$$= \det\left[r_{p2}^{k-q}\right]_{p,q=1,\ldots,k}\prod_{p=1}^{k}\frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}}$$
$$= \Delta_k(r_2)\prod_{p=1}^{k}\frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}}. \tag{C.8}$$

Collecting everything we arrive at the first of the results (59). The normalization constant c_{Nk} is found by integrating $I(\rho)$ together with a normalized Gaussian

$$C_{Nk}\left(\frac{(-1)^{N-1}(N-1)!}{\mathrm{i}2\pi}\right)^k = I(0) = 2^{k(k-1)}\int \mathrm{d}[\rho]\exp(-\mathrm{trg}\,\rho^2)I(\rho)$$
$$= \int \mathrm{d}[\sigma]\exp\left(-\frac{1}{4}\mathrm{trg}\,\sigma^2\right)\mathrm{detg}^{-N}\sigma^- = 2^{-k(k-1)}. \tag{C.9}$$

When calculating $I(0)$, we used $\Theta(0) = 1/2$, as follows from a careful inspection of the step leading from equation (C.1) to equation (C.2). The second and the last equality sign in equation (C.9) are due to the Efetov–Wegner–Parisi–Sourlas theorem [1, 22–25], and the equality of the two integrals is a direct consequence of the definition (58).

Appendix D. Properties of the fundamental correlations and the fundamental kernel

The fundamental correlations (73) are closely related, but not identical to the superdeterminant, i.e. to the Fourier backtransform of the function $I(\rho)$, defined in equation (58). This is so, because the Efetov–Wegner–Parisi–Sourlas contributions [1, 22–25] for the function $I(\rho)$ and for the whole generating function $Z_k(x+J)$ are different. To gain further insights, we rewrite the fundamental kernel (75). We introduce two 2×2 Hermitian supermatrices $\overline{\sigma}$ and $\overline{\rho}$ with eigenvalues $\overline{s} = \mathrm{diag}(s_{p1}, \mathrm{i}s_{q2})$ and $\overline{r} = \mathrm{diag}(r_1, \mathrm{i}r_2)$, respectively. We then can cast equation (75) into the form

$$\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) = -\frac{1}{\pi(s_{p1}^- - \mathrm{i}s_{q2})}\left(\int \mathrm{d}[\overline{\rho}]\exp(-\mathrm{i}\,\mathrm{trg}\,\overline{\rho\sigma})I(\overline{\rho}) - 1\right), \tag{D.1}$$

which is easily verified with the help of formula (70). Importantly, we have to subtract the Efetov–Wegner–Parisi–Sourlas contribution, i.e. unity in the present case, from the integral. The integral over $\overline{\rho}$ is now precisely the Fourier backtransform of equation (58) and we have

$$\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) = -\frac{1}{\pi(s_{p1}^- - \mathrm{i}s_{q2})}(\mathrm{detg}^{-N}\overline{\sigma}^- - 1)$$
$$= -\frac{1}{\pi(s_{p1}^- - \mathrm{i}s_{q2})}\left(\left(\frac{\mathrm{i}s_{q2}}{s_{p1}^-}\right)^N - 1\right)$$
$$= \frac{1}{\pi(s_{p1}^-)^N}\frac{(s_{p1}^-)^N - (\mathrm{i}s_{q2})^N}{s_{p1}^- - \mathrm{i}s_{q2}}, \tag{D.2}$$

which is the result (77). We note that the fundamental kernel and the fundamental correlation function coincide for $k = 1$. For $k > 1$, this is not so and the calculation just presented works

for the fundamental kernel, but not for the whole fundamental correlation function, because we do not know the corresponding Efetov–Wegner–Parisi–Sourlas contributions explicitly.

To calculate the double integral (75) directly, we apply techniques developed in [12, 26]. We integrate the r_2 integral $N-1$ times by parts and have

$$\widehat{C}^{(\mathrm{fund})}(s_{p1}, \mathrm{i}s_{q2}) = -\frac{(-1)^{N-1}}{\pi(N-1)!} \times \int_{-\infty}^{+\infty} \mathrm{d}r_1 \Theta(r_1)(\mathrm{i}r_1)^N \exp(-\mathrm{i}r_1 s_{p1}^-) \frac{\partial^{N-1}}{\partial r_2^{N-1}} \frac{\exp(-\mathrm{i}r_2 s_{q2})}{r_1 - \mathrm{i}r_2}\bigg|_{r_2=0}. \tag{D.3}$$

The $(N-1)$-fold derivative is computed with Leibnitz' rule according to

$$\frac{\partial^{N-1}}{\partial r_2^{N-1}} \frac{\exp(-\mathrm{i}r_2 s_{q2})}{r_1 - \mathrm{i}r_2}\bigg|_{r_2=0} = (-1)^N (N-1)! \sum_{n=0}^{N-1} \frac{(\mathrm{i}s_{q2})^n}{n!(\mathrm{i}r_1)^{N-n}}, \tag{D.4}$$

which yields equation (76).

Appendix E. Rederivation of the GUE correlation functions

Again, we resort to techniques developed in [12, 26]. For a Gaussian probability density $P(H) \sim \exp(-\mathrm{tr}\, H^2)$, we obviously have

$$\begin{aligned} P_1^{(\mathrm{red})}(H_{pp}) &= \frac{1}{\sqrt{\pi}} \exp\left(-H_{pp}^2\right) \\ P_2^{(\mathrm{red})}(H_{(k+p)(k+p)}) &= \frac{1}{\sqrt{\pi}} \exp\left(-H_{(k+p)(k+p)}^2\right) \end{aligned} \tag{E.1}$$

in equation (79). We plug this into equation (78) and use the first of equations (76). The $H_{(k+q)(k+q)}$, $q = 1, \ldots, k$ integrals are then recognized as representations of the Hermite polynomials

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\xi^2)(z - \mathrm{i}\xi)^n \,\mathrm{d}\xi, \tag{E.2}$$

while the H_{pp}, $p = 1, \ldots, k$ integrals yield the generalized Hermite functions

$$\widehat{H}_n(x) = \frac{(\mathrm{i}2)^{n+1}}{\sqrt{\pi}} \exp(x^2) \int_0^{\infty} \exp(-\xi^2 - \mathrm{i}2x\xi)\xi^n \,\mathrm{d}\xi. \tag{E.3}$$

These functions have already been introduced into the RMT context in [26]. They comprise both fundamental solutions of the Hermite differential equation, the polynomials $H_n(x)$ and the non-polynomial solutions $\widetilde{H}_n(x)$, in the form

$$\widehat{H}_n(x) = \widetilde{H}_n(x) + \mathrm{i}H_n(x), \tag{E.4}$$

details can be found in [26]. We also use the generalized oscillator wavefunctions

$$\widehat{\varphi}_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp\left(-\frac{x^2}{2}\right) \widehat{H}_n(x) = \widetilde{\varphi}_n(x) + \mathrm{i}\varphi_n(x). \tag{E.5}$$

Collecting everything, we find

$$\int_{-\infty}^{+\infty} \mathrm{d}H_{pp} \frac{1}{\sqrt{\pi}} \exp\left(-H_{pp}^2\right) \int_{-\infty}^{+\infty} \mathrm{d}H_{(k+q)(k+q)} \frac{1}{\sqrt{\pi}} \exp\left(-H_{(k+q)(k+q)}^2\right) \times \widehat{C}^{(\mathrm{fund})}(x_p - H_{pp}, x_q - \mathrm{i}H_{(k+q)(k+q)}) = \exp\left(\frac{x_q^2 - x_p^2}{2}\right) \widehat{K}^{(\mathrm{GUE})}(x_p, x_q), \tag{E.6}$$

where

$$\widehat{K}^{(\mathrm{GUE})}(x_p, x_q) = \sum_{n=0}^{N-1} \widehat{\varphi}_n(x_p)\varphi_n(x_q) \tag{E.7}$$

is the GUE kernel, including the real part correlations. Hence, we arrive at

$$\begin{aligned}\widehat{R}_k(x_1, \ldots, x_k) &= \det\left[\exp\left(\frac{x_q^2 - x_p^2}{2}\right)\widehat{K}^{(\mathrm{GUE})}(x_p, x_q)\right]_{p,q=1,\ldots,k} \\ &= \det[\widehat{K}^{(\mathrm{GUE})}(x_p, x_q)]_{p,q=1,\ldots,k} \\ &= \widehat{R}_k^{(\mathrm{GUE})}(x_1, \ldots, x_k),\end{aligned} \tag{E.8}$$

which is the correct result.

Appendix F. On the proper definition of the reduced probability density

The following example illustrates, despite its simplicity, why the definition of the reduced probability density $P^{(\mathrm{red})}(h)$ should involve the characteristic function. We consider the probability density

$$\begin{aligned}P(H) &= \frac{2^{N(N-1)/2}}{\pi^{N^2/2}} \exp(-\mathrm{tr}(H + \alpha 1_N)^2) \\ &= \frac{2^{N(N-1)/2}}{\pi^{N^2/2}} \exp(-\mathrm{tr}\, H^2 - 2\alpha\, \mathrm{tr}\, H - N\alpha^2),\end{aligned} \tag{F.1}$$

where α is a constant. The probability density $P(H)$ depends on two invariants, $\mathrm{tr}\, H$ and $\mathrm{tr}\, H^2$. The correlation functions are obviously those of the GUE, measured on the energy scale shifted by α,

$$\widehat{R}_k(x_1, \ldots, x_k) = \widehat{R}_k^{(\mathrm{GUE})}(x_1 + \alpha, \ldots, x_k + \alpha). \tag{F.2}$$

Formally, this follows directly from the definition (2) if one shifts the integration variables H by $\alpha 1_N$. The characteristic function reads

$$\begin{aligned}\Phi(K) &= \exp\left(-\tfrac{1}{4}\,\mathrm{tr}\, K^2 - \mathrm{i}\alpha\, \mathrm{tr}\, K\right) \\ &= \exp\left(-\tfrac{1}{4}\mathrm{trg}\, B^2 - \mathrm{i}\alpha\, \mathrm{trg}\, B\right) = \Phi(B)\end{aligned} \tag{F.3}$$

which yields with the replacement (35)

$$\Phi(r) = \exp\left(-\tfrac{1}{4}\,\mathrm{trg}\, r^2 - \mathrm{i}\alpha\, \mathrm{trg}\, r\right) = \exp\left(-\tfrac{1}{4}\,\mathrm{trg}(r + \mathrm{i}2\alpha 1_{2k})^2\right) \tag{F.4}$$

as function of the eigenvalues r. Since the characteristic function has the factorization property (96) with

$$\Phi_1^{(\mathrm{ev})}(r_{p1}) = \exp\left(-\frac{r_{p1}^2}{4} - \mathrm{i}\alpha r_{p1}\right) \qquad \Phi_2^{(\mathrm{ev})}(r_{p2}) = \exp\left(-\frac{r_{p2}^2}{4} - \alpha r_{p2}\right), \tag{F.5}$$

we can employ formulae (97) and (98) which directly give the correct result (F.2). Here, it is crucial that $\Phi_1^{(\mathrm{ev})}(r_{p1})$ and $\Phi_2^{(\mathrm{ev})}(r_{p2})$ are not the same functions, even though the difference is only due to an imaginary unit.

We now turn to the reduced probability density. From the definition (46) we find

$$\begin{aligned}P^{(\mathrm{red})}(h) &= \frac{1}{\pi^k} \exp\left(-\mathrm{trg}\, h^2 - 2\alpha\, \mathrm{trg}\, h\right) \\ &= \frac{1}{\pi^k} \exp(-\mathrm{trg}(h + \alpha 1_{2k})^2)\end{aligned} \tag{F.6}$$

implying the factorization (79) with

$$P_1^{(\mathrm{red})}(H_{pp}) = \frac{1}{\sqrt{\pi}} \exp(-(H_{pp}+\alpha)^2)$$
$$P_2^{(\mathrm{red})}(H_{(k+p)(k+p)}) = \frac{1}{\sqrt{\pi}} \exp(+(\mathrm{i}H_{(k+p)(k+p)}+\alpha)^2). \tag{F.7}$$

As this yields in formula (78) together with the results of appendix E once more the right answer (F.2) we have the confirmation that the reduced probability density (F.6) is indeed the correct one. On the other hand, the naive definition of the reduced probability density which we discussed in section 3.6 would give

$$\int \mathrm{d}[H/h]P(H) = \frac{1}{\pi^k} \prod_{p=1}^{k} \exp(-(H_{pp}+\alpha)^2) \prod_{p=1}^{k} \exp(-(H_{(k+p)(k+p)}+\alpha)^2), \tag{F.8}$$

which is easily seen to be wrong when inserted in formula (78). The correct reduced probability density (F.6) does contain information about supersymmetry, because the functions $P_1^{(\mathrm{red})}(H_{pp})$ and $P_2^{(\mathrm{red})}(H_{(k+p)(k+p)})$ are different. In the expression (F.8), however, the variables H_{pp} and $H_{(k+p)(k+p)}$ appear in exactly the same way, no traces of a supersymmetric structure are present.

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